

# The Complex Frobenius Theorem for Rough Involutive Structures

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**ABSTRACT.** We establish a version of the complex Frobenius theorem in the context of a complex subbundle  $\mathcal{S}$  of the complexified tangent bundle of a manifold, having minimal regularity. If the subbundle  $\mathcal{S}$  defines the structure of a Levi-flat CR-manifold, it suffices that  $\mathcal{S}$  be Lipschitz for our results to apply. A principal tool in the analysis is a precise version of the Newlander-Nirenberg theorem with parameters, for integrable almost complex structures with minimal regularity, which builds on previous recent work of the authors.

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## 1. Introduction

The complex Frobenius theorem elucidates the structure of a complex subbundle  $\mathcal{S}$  of the complexified tangent bundle  $\mathbb{C}T\Omega$  of a smooth manifold  $\Omega$ , satisfying an involutivity condition, which can be stated as follows: if  $X$  and  $Y$  are (sufficiently regular) sections of  $\mathcal{S}$ , then

$$(1.1) \quad [X, Y] \text{ is a section of } \mathcal{S},$$

and

$$(1.2) \quad [X, \overline{Y}] \text{ is a section of } \mathcal{S} + \overline{\mathcal{S}}.$$

Here, as usual, if  $X = X_0 + iX_1$  and  $X_0, X_1$  are real vector fields, we write  $\overline{X} = X_0 - iX_1$ , and the fiber of  $\overline{\mathcal{S}}$  over  $p \in \Omega$  is given as

$$(1.3) \quad \overline{\mathcal{S}}_p = \{u - iv : u + iv \in \mathcal{S}_p, \ u, v \in T_p\Omega\}.$$

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We also assume  $\mathcal{S} + \overline{\mathcal{S}}$  is a subbundle of  $\mathbb{C}T\Omega$ .

In case  $\mathcal{S} = \mathbb{C}\mathcal{S}_0$  is the complexification of a subbundle  $\mathcal{S}_0 \subset T\Omega$ , the condition (1.1) just says  $\mathcal{S}_0$  is involutive. (Here,  $\mathcal{S} = \overline{\mathcal{S}}$ , and (1.2) provides no additional constraint.) In this case the result reduces to the real Frobenius theorem.

An opposite extreme arises when  $\Omega$  has an almost complex structure, a section  $J$  of  $\text{End } T\Omega$  satisfying  $J^2 = -I$  (which implies that  $\dim \Omega$  is even). We set

$$(1.4) \quad \mathcal{S}_p = \{u + iJu : u \in T_p\Omega\},$$

so a section of  $\mathcal{S}$  has the form  $X + iJX$ , for a general real vector field  $X$ . The condition (1.1) is that if also  $Y$  is a real vector field, then  $[X + iJX, Y + iJY] = Z + iJZ$  for a real vector field  $Z$ . This is equivalent to the vanishing of the Nijenhuis tensor, defined by

$$(1.5) \quad N(X, Y) = [X, Y] - [JX, JY] + J[X, JY] + J[JX, Y].$$

The content of the Newlander-Nirenberg theorem [NN] is that under this formal integrability hypothesis  $\Omega$  has local holomorphic coordinates, i.e., functions  $u_1, \dots, u_k : \mathcal{O} \rightarrow \mathbb{C}$  forming a coordinate system on a neighborhood  $\mathcal{O}$  of a given  $p \in \Omega$ , such that  $(X + iJX)u_\ell \equiv 0$  for all real vector fields  $X$ . Thus  $\Omega$  has the structure of a complex manifold. In this case,  $\mathcal{S} + \overline{\mathcal{S}} = \mathbb{C}T\Omega$ , so (1.2) automatically holds. There are other cases, where (1.2) has a nontrivial effect, as will be seen below.

The complex Frobenius theorem was established in [Ni] for  $C^\infty$  bundles  $\mathcal{S} \subset \mathbb{C}T\Omega$  satisfying (1.1)–(1.2). A major ingredient in the proof was the Newlander-Nirenberg theorem, which had been established in [NN] for almost complex structures with a fairly high degree of smoothness. Later proofs of the Newlander-Nirenberg theorem, by [NW] and by [M], work for almost complex structures  $J$  of class  $C^{1+r}$  with  $r > 0$ , i.e., when  $J$  has Hölder continuous first order derivatives. In [HT] the needed regularity on  $J$  was reduced to  $J \in C^r$  with  $r > 1/2$ . (More general conditions were considered in [HT], which we will not discuss here.) The case of Lipschitz  $J$  found an immediate application in [LM].

Regarding the real Frobenius theorem, standard arguments, though frequently phrased in the context of smooth subbundles of  $T\Omega$ , work for  $C^1$  bundles. The real Frobenius theorem was extended in [Ha] to include Lipschitz subbundles.

Our main goal here is to extend Nirenberg's complex Frobenius theorem to the setting of rough bundles  $\mathcal{S} \subset \mathbb{C}T\Omega$  satisfying (1.1)–(1.2). We will assume that  $\mathcal{S}$  and  $\mathcal{S} + \overline{\mathcal{S}}$  are Lipschitz subbundles of  $\mathbb{C}T\Omega$ . Note that if  $X$  and  $Y$  are Lipschitz sections of  $\mathcal{S}$ , then  $[X, Y]$  and  $[X, \overline{Y}]$  are vector fields with  $L^\infty$  coefficients. For an important class of bundles  $\mathcal{S}$ , namely those giving rise to Levi-flat CR-structures (defined below) this regularity hypothesis will suffice. In the general case we need an additional hypothesis, given in (1.16) below. We mention that [Ho] established a version of a complex Frobenius theorem in a setting of  $C^1$  vector fields, with  $C^1$  commutators, but with a somewhat different thrust.

We now set up a basic strategy for obtaining such a complex Frobenius theorem, and indicate what extra analysis has to be done to treat the non-smooth case. It is convenient to begin by constructing some further subbundles of the real tangent bundle  $T\Omega$ . For each  $p \in \Omega$ , set

$$(1.6) \quad \begin{aligned} \mathcal{E}_p &= \{u \in T_p\Omega : u + iv \in \mathcal{S}_p, \text{ for some } v \in T_p\Omega\} \\ &= \{w + \overline{w} : w \in \mathcal{S}_p\}, \end{aligned}$$

the fiber over  $p$  of a Lipschitz bundle  $\mathcal{E}$ . Noting that if  $u, v \in T_p\Omega$  and  $u + iv \in \mathcal{S}_p$ , then also  $v - iu \in \mathcal{S}_p$ , so  $v \in \mathcal{E}_p$ , we see that

$$(1.7) \quad \mathcal{S} + \overline{\mathcal{S}} = \mathbb{C}\mathcal{E}.$$

Next, set

$$(1.8) \quad \mathcal{V}_p = \mathcal{S}_p \cap T_p\Omega,$$

the fiber over  $p$  of a Lipschitz vector bundle  $\mathcal{V}$ . Note that if  $u, v \in T_p\Omega$ ,

$$(1.9) \quad \begin{aligned} u + iv \in \mathcal{S}_p \cap \overline{\mathcal{S}}_p &\iff u + iv \in \mathcal{S}_p \text{ and } u - iv \in \mathcal{S}_p \\ &\iff u \in \mathcal{S}_p \text{ and } v \in \mathcal{S}_p. \end{aligned}$$

Hence

$$(1.10) \quad \mathcal{S} \cap \overline{\mathcal{S}} = \mathbb{C}\mathcal{V}.$$

The hypotheses (1.1)–(1.2) imply  $\mathcal{E}$  and  $\mathcal{V}$  are involutive subbundles of  $T\Omega$ , i.e.,

$$(1.11) \quad \begin{aligned} X, Y \in \text{Lip}(\Omega, \mathcal{E}) &\implies [X, Y] \in L^\infty(\Omega, \mathcal{E}), \\ X, Y \in \text{Lip}(\Omega, \mathcal{V}) &\implies [X, Y] \in L^\infty(\Omega, \mathcal{V}). \end{aligned}$$

On the other hand, one does not recover (1.1)–(1.2) from (1.11) alone, as our second example illustrates. In that example, with  $\mathcal{S}_p$  given by (1.4), we have  $\mathcal{E} = T\Omega$ ,  $\mathcal{V} = 0$ , and (1.11) always holds, regardless of whether  $N$  in (1.5) vanishes. To capture (1.1)–(1.2), an additional structure arises.

Namely, one has a complex structure on the quotient bundle  $\mathcal{E}/\mathcal{V}$ , defined as follows. Take  $u \in \mathcal{E}_p$ , so there exists  $v \in T_p\Omega$  such that  $u + iv \in \mathcal{S}_p$ ; in fact,  $v \in \mathcal{E}_p$ . We propose to set  $Ju = v$ , so the element of  $\mathcal{S}_p$  has the form  $u + iJu$ . However, the element  $v$  associated to  $u \in \mathcal{E}_p$  is not necessarily unique. In fact, given  $u, v, v' \in T_p\Omega$  and  $u + iv \in \mathcal{S}_p$ , we have

$$(1.12) \quad u + iv' \in \mathcal{S}_p \iff i(v - v') \in \mathcal{S}_p \iff v - v' \in \mathcal{S}_p \cap \mathcal{E}_p = \mathcal{V}_p.$$

In other words, given  $u \in \mathcal{E}_p$ , the residue class of  $Ju$  is well defined in  $\mathcal{E}_p/\mathcal{V}_p$ . Furthermore, if  $u \in \mathcal{V}_p$ , one can take  $v = 0$ , so  $J$  descends from a linear map  $\mathcal{E}_p \rightarrow \mathcal{E}_p/\mathcal{V}_p$  to

$$(1.13) \quad J_p : \mathcal{E}_p/\mathcal{V}_p \longrightarrow \mathcal{E}_p/\mathcal{V}_p,$$

yielding

$$(1.14) \quad J \in \text{Lip}(\Omega, \text{End } \mathcal{E}/\mathcal{V}).$$

Since  $u+iv \in \mathcal{S}_p \Leftrightarrow v-iu \in \mathcal{S}_p$ , we also have  $J^2 = -I$ . The integrability hypotheses (1.1)–(1.2) are equivalent to (1.11), coupled to an integrability hypothesis on  $J$ , which we describe below.

Let us first consider the case  $\mathcal{V} = 0$ . Then  $J$  is a complex structure on the involutive bundle  $\mathcal{E}$ , and (generalizing (1.4)) we have

$$(1.15) \quad \mathcal{S}_p = \{u + iJu : u \in \mathcal{E}_p\},$$

or equivalently Lipschitz sections of  $\mathcal{S}$  have the form  $X + iJX$ , where  $X$  is a Lipschitz section of  $\mathcal{E}$ . Then the involutivity hypothesis (1.1)–(1.2) is equivalent to the involutivity of  $\mathcal{E}$  plus the vanishing of  $N$ , given by (1.5), for  $X, Y \in \text{Lip}(\Omega, \mathcal{E})$ . One says that  $\Omega$  has the structure of a Levi-flat CR manifold. The real Frobenius theorem implies that  $\Omega$  is foliated by leaves tangent to  $\mathcal{E}$ . Each such leaf then inherits an almost complex structure, and the Newlander-Nirenberg theorem implies each such leaf has local holomorphic coordinates. Briefly put,  $\Omega$  is foliated by complex manifolds. The complex Frobenius theorem in this context says a little more. Namely, any  $p \in \Omega$  has a neighborhood  $\mathcal{O}$  on which there are functions  $u_1, \dots, u_k$ , providing holomorphic coordinates on each leaf, intersected with  $\mathcal{O}$ , and having some regularity on  $\mathcal{O}$ . In the case of a  $C^\infty$  bundle  $\mathcal{S}$ , [Ni] obtained such  $u_j \in C^\infty(\mathcal{O})$ . In the context of Lipschitz structures, we obtain certain Hölder continuity of  $u_j$ , described in further detail below. A key ingredient in the analysis is a Newlander-Nirenberg theorem with parameters. In the smooth case this follows by the methods of [NN], as noted there and used in [Ni]. We devote §4 to a consideration of families of integrable almost complex structures with minimal regularity, building on techniques of [M] and of [HT].

We now turn to the case  $\mathcal{V} \neq 0$ . In this case, we supplement the Lipschitz hypotheses on  $\mathcal{S}$  and  $\mathcal{S} + \overline{\mathcal{S}}$  with the following hypothesis. Say  $\dim \mathcal{V}_p = \ell \leq k = \dim \mathcal{E}_p$ . We assume that each  $p \in \Omega$  has a neighborhood on which there is a local Lipschitz frame field  $\{X_1, \dots, X_k\}$  for  $\mathcal{E}$ , such that  $\{X_1, \dots, X_\ell\}$  is a local frame field for  $\mathcal{V}$  and

$$(1.16) \quad [X_i, X_j] = 0, \quad 1 \leq i, j \leq k.$$

This can be regarded as an hypothesis on the regularity with which  $\mathcal{V}$  sits in  $\mathcal{E}$ ; we discuss it further in §6. We will show that

$$(1.17) \quad J \text{ is invariant under } \mathcal{F}_{X_i}^t, \quad 1 \leq i \leq \ell,$$

where  $\mathcal{F}_{X_i}^t$  is the flow generated by  $X_i$ . Hence we can mod out by the  $\mathcal{F}_{X_1}^{t_1} \circ \cdots \circ \mathcal{F}_{X_\ell}^{t_\ell}$  action, to obtain

$$(1.18) \quad \pi : \Omega \longrightarrow M,$$

(perhaps after localizing), and on  $M$  we have a Levi-flat CR structure. Leafwise holomorphic functions on (open subsets of)  $M$  pull back to functions on (open subsets of)  $\Omega$ , and results on their existence and regularity essentially constitute the complex Frobenius theorem for the bundle  $\mathcal{S}$ .

The rest of this paper is organized as follows. Section 2 treats the real Frobenius theorem for involutive Lipschitz bundles. We recall some results of [Ha] and establish some further results, regarding the regularity of the diffeomorphism constructed to flatten out the leaves of the foliation. In §3 we consider Levi-flat CR manifolds, in the Lipschitz category, even allowing for rougher  $J$ , and examine how such a structure pulls back under a leaf-flattening diffeomorphism from §2, to yield a parametrized family of manifolds carrying integrable almost complex structures. This sets us up for a study of the Newlander-Nirenberg theorem with parameters, which we carry out in §4.

In §5 we tie together the material of §§2–4 to obtain results on the existence and regularity of functions on open sets of a Lipschitz Levi-flat CR manifold  $\Omega$  that are leafwise holomorphic (functions known as CR functions). Our primary result, Proposition 5.1, yields CR functions  $\varphi_j$ ,  $1 \leq j \leq m+n-k$ , on a neighborhood  $U_1$  of a point  $p \in \Omega$ , having the property that

$$(1.19) \quad \Phi = (\varphi_1, \dots, \varphi_{m+n-k}) : U_1 \longrightarrow \mathbb{C}^m \times \mathbb{R}^{n-k}$$

is a homeomorphism of  $U_1$  onto an open subset, and such that, given  $s < 1/2$ ,  $\varphi_j$  and  $X\varphi_j$  are Hölder continuous of degree  $s$ , for any  $X \in \text{Lip}(U_1, \mathcal{E})$ . A complementary result, Proposition 5.2, shows that  $\Phi$  in (1.19) can be taken to be a  $C^1$  diffeomorphism, provided that  $\mathcal{S}$ , and hence  $\mathcal{E}$  and  $J$ , are regular of class  $C^\rho$  for some  $\rho > 3/2$ . The results of [HT] extending the Newlander-Nirenberg theorem to cases where the almost complex structure is merely  $C^{1/2+\varepsilon}$  regular, and complementary results of §4, play an important role in the proof. We end §5 with a brief discussion of  $C^{1,1}$  submanifolds of  $\mathbb{C}^N$  that have the structure of Levi-flat CR-manifolds. The general complex Frobenius theorem is then treated in §6.

At the end of this paper we have two appendices. Appendix A is devoted to a Frobenius theorem for real analytic, complex vector fields. There are classical results of this nature; cf. [Ni] for some references. One motivation for us to include a

self contained treatment of such a result here arises from the nature of our analysis of the Newlander-Nirenberg theorem with parameters in §4. Following [M], we construct the local holomorphic coordinate chart as a composition,  $F = G \circ H$ . The map  $H$  is obtained via an implicit function theorem, the use of which enables us to keep track of its dependence on a parametrized family of integrable almost complex structures. The construction of  $H$  arranges things so that constructing  $G$  amounts to establishing the Newlander-Nirenberg theorem in the real analytic category, a task to which the material of Appendix A is applicable, and this material makes it clear how the factor  $G$  depends on parameters.

Finally, Appendix B gives a special treatment of the construction of CR functions on a rough Levi-flat CR manifold whose leaves have real dimension 2. The classical method of constructing isothermal coordinates is adapted to this problem and yields sharper results than one obtains in the case of higher dimensional leaves via the methods of §4. This leads to improved results in §5 in the case of 2-dimensional leaves, as is noted there.

We end this introduction with a few remarks on function spaces arising in our analysis. For a smoothly bounded domain  $\overline{U}$ ,  $C^r(\overline{U})$  denotes the space of functions with derivatives of order  $\leq r$  continuous on  $\overline{U}$ , if  $r$  is a positive integer. If  $r = k + s$ ,  $k \in \mathbb{Z}^+$ ,  $0 < s < 1$ , it denotes the space of functions whose  $k$ th order derivatives are Hölder continuous of order  $s$ . In addition, we make use of Zygmund spaces  $C_*^r(\overline{U})$ , coinciding with  $C^r(\overline{U})$  for  $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$ , and having nice interpolation properties at  $r \in \mathbb{Z}^+$ . The spaces  $C_*^r(\overline{U})$  are also defined for  $r < 0$ . There are a number of available treatments of Zygmund spaces; we mention Chapter 13, §8 of [T] as one source. As is usual,  $\text{Lip}(\overline{U})$  denotes the space of Lipschitz continuous functions, i.e., functions Hölder continuous of exponent one, and  $C^{1,1}(\overline{U})$  denotes the space of functions whose first order derivatives belong to  $\text{Lip}(\overline{U})$ .

## 2. Real Frobenius theorem for involutive Lipschitz bundles

Let  $\mathcal{E}$  be a sub-bundle of the tangent bundle  $T\Omega$ , of fiber dimension  $k$ . We assume  $\mathcal{E}$  is Lipschitz, in the sense that any  $p_0 \in \Omega$  has a neighborhood  $\mathcal{O}$  on which there are Lipschitz vector fields  $X_1, \dots, X_k$  spanning  $\mathcal{E}$  at each point. We make the involutivity hypothesis that  $[X_i, X_j]$  is a section of  $\mathcal{E}$  at almost all points of  $\mathcal{O}$ , or equivalently that there exist  $c_{ij}^\ell \in L^\infty(\mathcal{O})$  such that

$$(2.1) \quad [X_i, X_j] = \sum_{\ell} c_{ij}^\ell(x) X_\ell.$$

We want to discuss the existence and qualitative properties of the foliation of  $\Omega$  whose leaves are tangent to  $\mathcal{E}$ .

We may as well assume  $k < n = \dim \Omega$ . Suppose we have coordinates centered at  $p_0$  such that  $X_j(p_0)$  form the first  $k$  standard basis elements of  $\mathbb{R}^n$ , for  $1 \leq j \leq k$ .

If we denote by  $\tilde{X}_j(x)$  the image of  $X_j(x)$  under the standard projection  $\mathbb{R}^n \rightarrow \mathbb{R}^k$ , we have

$$(2.2) \quad \tilde{X}_i(x) = \sum_{j=1}^k A_{ij}(x) \partial_j,$$

with  $A_{ij} \in \text{Lip}(\mathcal{O})$ ,  $A_{ij}(p_0) = \delta_{ij}$ , hence  $(A_{ij}(x))$  an invertible  $k \times k$  matrix, with inverse  $(B_{ij}(x))$ , for  $x$  in a neighborhood of  $p_0$  (which we now denote  $\mathcal{O}$ ). We set

$$(2.3) \quad Y_i = \sum_j B_{ij}(x) X_j, \quad 1 \leq i \leq k.$$

It follows that

$$(2.4) \quad Y_i = \partial_i + Y_i^\#, \quad Y_i^\# = \sum_{\ell \geq k+1} D_{i\ell}(x) \partial_\ell, \quad 1 \leq i \leq k.$$

Also (2.1) implies

$$(2.5) \quad [Y_i, Y_j] = \sum_\ell \tilde{c}_{ij}^\ell(x) Y_\ell,$$

for certain  $\tilde{c}_{ij}^\ell \in L^\infty(\mathcal{O})$ . Comparison of (2.4) and (2.5) yields  $\tilde{c}_{ij}^\ell \equiv 0$ , so we have a local Lipschitz frame field for  $\mathcal{E}$  satisfying

$$(2.6) \quad [Y_i, Y_j] = 0, \quad 1 \leq i, j \leq k.$$

The key result on the existence of a foliation tangent to  $\mathcal{E}$  is the following result of [Ha].

**Proposition 2.1.** *Let  $Y_j$  be Lipschitz vector fields on  $\mathcal{O}$  satisfying (2.6). For any compact  $K \subset \mathcal{O}$  there exists  $\delta > 0$  such that there is a unique solution  $y = y(t, x_0) = y(t_1, \dots, t_k, x_0)$  to*

$$(2.7) \quad \frac{\partial y}{\partial t_j} = Y_j(y), \quad 1 \leq j \leq k, \quad y(0, x_0) = x_0,$$

given  $x_0 \in K$ ,  $|t_j| < \delta$ . Furthermore,  $y(t, x)$  is Lipschitz in  $(t, x)$ .

In fact, this result is a special case of Corollary 4.1 of [Ha]. We make some further comments on it. If  $\mathcal{F}_{Y_j}^t$  denotes the flow generated by  $Y_j$ , we see that

$$(2.8) \quad y(0, \dots, 0, t_k, x) = \mathcal{F}_{Y_k}^{t_k}(x).$$

Then

$$(2.9) \quad y(0, \dots, 0, t_{k-1}, t_k, x) = \mathcal{F}_{Y_{k-1}}^{t_{k-1}} \circ \mathcal{F}_{Y_k}^{t_k}(x),$$

and inductively

$$(2.10) \quad y(t_1, \dots, t_k, x) = \mathcal{F}_{Y_1}^{t_1} \circ \dots \circ \mathcal{F}_{Y_k}^{t_k}(x).$$

The order can be changed, and we have

$$(2.11) \quad \mathcal{F}_{Y_i}^{t_i} \circ \mathcal{F}_{Y_j}^{t_j}(x) = \mathcal{F}_{Y_j}^{t_j} \circ \mathcal{F}_{Y_i}^{t_i}(x),$$

for  $x \in K$ ,  $|t_i|, |t_j| < \delta$ .

Conversely, once one knows that (2.11) follows from (2.6), one can prove Proposition 2.1. However, this implication is less straightforward for Lipschitz vector fields than it is for smooth vector fields. In connection with this, we mention the following analytical point, which plays a key role in the proof in [Ha]. Namely, let  $\{J_\varepsilon : 0 < \varepsilon \leq 1\}$  be a Friedrichs mollifier and let  $Y_i, Y_j$  be Lipschitz vector fields satisfying (2.6). Then, as  $\varepsilon \rightarrow 0$ ,

$$(2.12) \quad [J_\varepsilon Y_i, J_\varepsilon Y_j] \longrightarrow 0,$$

locally uniformly on  $\mathcal{O}$ . Actually this is a reformulation (of a special case) of Proposition 5.3 of [Ha]. It is stronger and more useful than the obvious fact that such convergence holds weak\* in  $L^\infty$ . What is behind it is the more general fact that, for any two Lipschitz vector fields  $X$  and  $Y$  on  $\mathcal{O}$ ,

$$(2.13) \quad [J_\varepsilon X, J_\varepsilon Y] - J_\varepsilon[X, Y] \longrightarrow 0,$$

locally uniformly on  $\mathcal{O}$ . This follows from the fact that

$$(2.14) \quad f \in \text{Lip}(\mathcal{O}), g \in L^\infty(\mathcal{O}) \implies (J_\varepsilon f)(J_\varepsilon g) - J_\varepsilon(fg) \rightarrow 0,$$

locally uniformly on  $\mathcal{O}$ , and since clearly  $J_\varepsilon f \rightarrow f$  locally uniformly on  $\mathcal{O}$  this in turn is equivalent to the fact that

$$(2.15) \quad f \in \text{Lip}(\mathcal{O}), g \in L^\infty(\mathcal{O}) \implies f J_\varepsilon g - J_\varepsilon(fg) \rightarrow 0,$$

locally uniformly on  $\mathcal{O}$ , which is a standard Friedrichs-type commutator estimate.

We record that  $y(t, x)$  has extra regularity in  $t$ .



**Corollary 2.2.** *For each  $j \in \{1, \dots, k\}$ ,*

$$(2.16) \quad \frac{\partial}{\partial t_j} y(t, x) \text{ is Lipschitz in } (t, x).$$

*Proof.* Clearly the right side of (2.7) is Lipschitz in  $(t, x)$ .

Recall that we are in a coordinate system in which (2.4) holds, with  $Y_i^\#(p_0) = 0$ ,  $p_0 = 0$ . For  $z$  close to 0 in  $\mathbb{R}^{n-k}$  and  $|t| < \delta$ , we define

$$(2.17) \quad G(t, z) = y(t, 0, z) = \mathcal{F}^t(0, z),$$

where we set

$$(2.18) \quad \mathcal{F}^t = \mathcal{F}_{Y_1}^{t_1} \circ \dots \circ \mathcal{F}_{Y_k}^{t_k}.$$

**Proposition 2.3.** *There is a neighborhood  $U_0$  of  $(0, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$  and a neighborhood  $U_1$  of  $p_0 \in \mathcal{O}$  such that*

$$(2.19) \quad G : U_0 \longrightarrow U_1$$

*is a Lipschitz homeomorphism, with Lipschitz inverse.*

*Proof.* We want to show that if  $(t, z)$  and  $(s, w)$  are distinct points in a small neighborhood of  $(0, 0)$ , then  $x_1 = G(t, z)$  and  $x_2 = G(s, w)$  are not too close. Note that

$$(2.20) \quad \mathcal{F}^{-t}(x_1) = (0, z), \quad \mathcal{F}^{-t}(x_2) = \mathcal{F}^{s-t}(0, w).$$

Since  $\mathcal{F}^{-t}$  is Lipschitz, we have

$$(2.21) \quad |\mathcal{F}^{-t}(x_1) - \mathcal{F}^{-t}(x_2)| \leq C|x_1 - x_2|.$$

Meanwhile, since the span of  $Y_1, \dots, Y_k$  is transversal to  $\{(0, z)\}$  near  $(0, z) = (0, 0)$ , we have

$$(2.22) \quad |(0, z) - \mathcal{F}^{s-t}(0, w)| \geq C(|z - w| + |s - t|).$$

Comparing (2.20)–(2.22) yields

$$(2.23) \quad |x_1 - x_2| \geq C(|z - w| + |s - t|),$$

as desired.

### 3. The pull-back of a Levi-flat CR structure

In §2 we constructed a bi-Lipschitz map

$$(3.1) \quad G : U_0 \longrightarrow U_1, \quad G(t, z) = \mathcal{F}^t(0, z),$$

taking sets  $z = z_0$  to leaves of the foliation whose tangent space is the involutive Lipschitz bundle  $\mathcal{E} \subset TU_1$ . Let us denote by  $\mathcal{E}_0 \subset TU_0$  the pull-back of  $\mathcal{E}$ , so  $\mathcal{E}_0$  is spanned by  $\partial/\partial t_j$ ,  $1 \leq j \leq k$ . Now we take  $k = 2m$  and suppose there is a complex structure on  $\mathcal{E}$ ,  $J \in \text{End}(\mathcal{E})$ . We pull this back to a complex structure  $J^0 \in \text{End}(\mathcal{E}_0)$ , examine its regularity, and show that if  $J$  is formally integrable then so is  $J^0$ .

Since Lipschitz sections of  $\mathcal{E}$  are given as linear combinations over  $\text{Lip}(U_1)$  of the vector fields  $Y_1, \dots, Y_k$ , the action of  $J$  is given by

$$(3.2) \quad JY_i = \sum_{j=1}^k \mathcal{J}_{ij}(x) Y_j.$$

We can make various hypotheses on the regularity of  $J$ . For example, we might assume

$$(3.3) \quad \mathcal{J}_{ij} \in \text{Lip}(U_1),$$

or we might make the weaker hypothesis

$$(3.4) \quad \mathcal{J}_{ij} \in C^r(U_1),$$

for some  $r \in (1/2, 1)$ . In any case, the complex structure induced on  $\mathcal{E}_0$  is given by

$$(3.5) \quad J^0 \frac{\partial}{\partial t_i} = \sum_{j=1}^k \mathcal{J}_{ij}^0(t, z) \frac{\partial}{\partial t_j}, \quad \mathcal{J}_{ij}^0(t, z) = \mathcal{J}_{ij}(G(t, z)).$$

It is clear that

$$(3.6) \quad \begin{aligned} \mathcal{J}_{ij} \in \text{Lip}(U_1) &\implies \mathcal{J}_{ij}^0 \in \text{Lip}(U_0), \\ \mathcal{J}_{ij} \in C^r(U_1) &\implies \mathcal{J}_{ij}^0 \in C^r(U_0), \end{aligned}$$

the latter provided  $0 < r < 1$ .

We next discuss integrability conditions. One approach would be to form the “Nijenhuis” tensor, associated to  $J$  by

$$(3.7) \quad N(X, Y) = [X, Y] - [JX, JY] + J[X, JY] + J[JX, Y],$$

for Lipschitz sections  $X$  and  $Y$  of  $\mathcal{E}$ . If  $J$  is Lipschitz, then (3.7) belongs to  $L^\infty(U_1)$ . If  $J$  satisfies (3.4) with  $r > 1/2$ , then by Lemma 1.2 of [HT], the right side of (3.7) is a distribution, belonging to  $C_*^{r-1}(U_1)$ . Now such a singular distribution does not necessarily pull back well under a bi-Lipschitz map. Instead, we will work on individual leaves.

We start by defining  $\mathcal{N}_{z_0}^0$ , associated with  $J^0$ , on a leaf in  $(t, z)$ -space where  $z = z_0$  is constant. We set

$$(3.8) \quad \mathcal{N}_{z_0}^0(X, Y) = [X, Y] - [J^0 X, J^0 Y] + J^0[X, J^0 Y] + J^0[J^0 X, Y],$$

where  $X$  and  $Y$  are linear combinations of  $\partial/\partial t_i$ ,  $1 \leq i \leq k$ , and  $J^0 = J^0(t, z_0)$ . For each fixed  $z_0$ , this defines an element of  $L^\infty(\mathcal{O}_0)$  if  $J^0$  is Lipschitz and an element of  $C_*^{r-1}(\mathcal{O}_0)$  if (3.4) holds, with  $r \in (1/2, 1)$ . Here  $\mathcal{O}_0 = \{t \in \mathbb{R}^k : |t| < \delta\}$ . As we have seen, for each  $z_0$  close to 0,  $G_{z_0}(t) = G(t, z_0)$  yields a  $C^{1,1}$ -diffeomorphism of  $\mathcal{O}_0$  onto a neighborhood of  $x_0 = G(0, z_0)$  in the leaf through  $x_0$ . In light of this, the following is useful.

**Proposition 3.1.** *Assume  $\varphi : \mathcal{O}_0 \rightarrow \mathcal{O}_1$  is a  $C^{1,1}$ -diffeomorphism between open sets in  $\mathbb{R}^k$ . Then the pull-back*

$$(3.9) \quad \varphi^* : \text{Lip}(\mathcal{O}_1) \rightarrow \text{Lip}(\mathcal{O}_0), \quad \varphi^* f(x) = f(\varphi(x))$$

*extends to*

$$(3.10) \quad \varphi^* : H^{s,p}(\mathcal{O}_1) \longrightarrow H^{s,p}(\mathcal{O}_0),$$

*for each  $s \in [-1, 1]$ ,  $p \in (1, \infty)$ . Furthermore, for each  $r \in (0, 1)$ ,*

$$(3.11) \quad \varphi^* : C_*^{r-1}(\mathcal{O}_1) \longrightarrow C_*^{r-1}(\mathcal{O}_0).$$

*Proof.* The result (3.10) is easy for  $s = 0, 1$ , and follows by interpolation for  $s \in (0, 1)$ . Now suppose  $s \in [-1, 0)$ . We have, for compactly supported  $u$ ,

$$(3.12) \quad \begin{aligned} (u, \varphi^* v) &= \int u(x) v(\varphi(x)) dx \\ &= \int u(\varphi^{-1}(x)) v(x) |\det D\varphi^{-1}(x)| dx. \end{aligned}$$

We have  $|\det D\varphi^{-1}| \in \text{Lip}(\mathcal{O}_1)$ , hence (by the case already treated)  $u \in H^{\sigma,q} \Rightarrow (u \circ \varphi)|\det D\varphi^{-1}| \in H^{\sigma,q}$ , for  $\sigma \in [0,1], q \in (1,\infty)$ . Thus by duality we have  $v \in H^{s,p} \Rightarrow \varphi^*v \in H^{s,p}$ , for  $s \in [-1,0], p \in (1,\infty)$ , as desired. Next, note that if  $g \in \text{Lip}(\mathcal{O}_1)$  and  $X$  is a Lipschitz vector field on  $\mathcal{O}_1$ , then  $\varphi$  transforms  $X$  to a Lipschitz vector field  $\tilde{X}$  on  $\mathcal{O}_0$  and

$$(3.13) \quad \varphi^*(Xg) = \tilde{X}\varphi^*g,$$

as elements of  $L^\infty(\mathcal{O}_0)$ . Now if  $g \in C^r(\mathcal{O}_1)$ , we have  $\varphi^*g \in C^r(\mathcal{O}_0)$  and then  $\tilde{X}\varphi^*g \in C_*^{r-1}(\mathcal{O}_0)$ , which yields (3.11).

REMARK. More generally, if  $\varphi$  is a diffeomorphism of class  $C^{1+r}$ ,  $r \in (0,1)$ , then (3.13) holds with  $\tilde{X}$  a  $C^r$ -vector field. Also, by Lemma 1.2 of [HT],

$$(3.14) \quad \tilde{g} \in C^r \implies \tilde{X}\tilde{g} \in C_*^{r-1}, \quad \text{provided } r > \frac{1}{2},$$

so (3.11) still holds, as long as  $r > 1/2$ .

#### 4. The Newlander-Nirenberg theorem with parameters

The Newlander-Nirenberg theorem provides local holomorphic coordinates on a manifold  $\Omega$  with an almost complex structure satisfying the formal integrability condition that its Nijenhuis tensor vanishes. In the setting of a relatively smooth almost complex structure  $J$  the smooth dependence of such coordinate functions on  $J$  was noted in [NN] and played a role in [Ni]. Here we aim to examine the dependence of such coordinates on  $J$ , in appropriate function spaces, in the context of the lower regularity hypotheses made here. Verifying this regularity will involve giving a review of the method of construction of holomorphic coordinates introduced in [M], with modifications as in [HT] to handle the still weaker regularity hypotheses made here.

Given  $p_0 \in \Omega$ , take coordinates  $x = (x_1, \dots, x_{2m})$ , centered at  $p_0$ , with respect to which

$$(4.1) \quad J(p_0)\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_{j+m}}, \quad J(p_0)\frac{\partial}{\partial x_{j+m}} = -\frac{\partial}{\partial x_j}, \quad 1 \leq j \leq m.$$

The condition for a function  $f$ , defined near  $p_0$ , to be holomorphic, is that  $f$  be annihilated by the vector fields

$$(4.2) \quad X_j = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + iJ\frac{\partial}{\partial x_j}\right), \quad 1 \leq j \leq m,$$

and in light of (4.1) we have  $J(\partial/\partial x_j) = \partial/\partial x_{j+m} + \sum_{\ell=1}^{2m} c_{j\ell} \partial/\partial x_\ell$  with  $c_{j\ell}(0) = 0$  ( $p_0 = 0$ ). Setting  $y_j = x_{j+m}$ ,  $\partial/\partial z_j = (1/2)(\partial/\partial x_j - i\partial/\partial y_j)$ ,  $\partial/\partial \bar{z}_j = (1/2)(\partial/\partial x_j + i\partial/\partial y_j)$ , we can write these complex vector fields as

$$(4.3) \quad \frac{\partial}{\partial \bar{z}_j} + \sum_{\ell=1}^m \left( \alpha_{j\ell} \frac{\partial}{\partial \bar{z}_\ell} + \beta_{j\ell} \frac{\partial}{\partial z_\ell} \right), \quad 1 \leq j \leq m.$$

Next, by a device similar to that used in (2.2)–(2.4), we can take linear combinations of these vector fields to obtain

$$(4.4) \quad Z_j = \frac{\partial}{\partial \bar{z}_j} - \sum_{\ell=1}^m a_{j\ell} \frac{\partial}{\partial z_\ell}, \quad 1 \leq j \leq m.$$

If  $J$  is of class  $C^r$ , then the coefficients in (4.3) and (4.4) are also of class  $C^r$ .

The formal integrability condition is that the Lie brackets  $[X_j, X_\ell]$  are all linear combinations of  $X_1, \dots, X_m$ . If  $J \in C^1$ , then  $[X_j, X_\ell]$  is a linear combination with continuous coefficients. If  $J \in C^r$  with  $r > 1/2$ , then the Lie brackets are still well defined, and the coefficients are distributions of class  $C_*^{r-1}$ . In such a case, it follows that the brackets  $[Z_j, Z_\ell]$  are linear combinations of  $Z_1, \dots, Z_m$ , which forces

$$(4.5) \quad [Z_j, Z_\ell] = 0, \quad 1 \leq j, \ell \leq m.$$

It is convenient to use matrix notation. Set  $A_j = (a_{j1}, \dots, a_{jm})$  (a row vector),  $A = (a_{j\ell})$ ,  $F = (f_1, \dots, f_m)$  (a row vector), and  $\partial/\partial \bar{z} = (\partial/\partial \bar{z}_1, \dots, \partial/\partial \bar{z}_m)^t$  (a column vector). The condition that  $f_1, \dots, f_m$  be  $J$ -holomorphic is that

$$(4.6) \quad \frac{\partial F}{\partial \bar{z}} = A \frac{\partial F}{\partial z},$$

and the formal integrability condition (4.5) is

$$(4.7) \quad \frac{\partial A_j}{\partial \bar{z}_\ell} + A_j \frac{\partial A_\ell}{\partial z} = \frac{\partial A_\ell}{\partial \bar{z}_j} + A_\ell \frac{\partial A_j}{\partial z}, \quad 1 \leq j, \ell \leq m.$$

The proof of the Newlander-Nirenberg theorem consists of the construction of  $F$ , mapping a neighborhood of  $p_0$  in  $\Omega$  diffeomorphically onto a neighborhood of 0 in  $\mathbb{C}^m$ , and solving (4.6).

Malgrange's method constructs  $F$  as a composition

$$(4.8) \quad F = G \circ H.$$

Different techniques are applied to construct the diffeomorphisms  $G$  and  $H$ . We run through these constructions, paying particular attention to the dependence on the matrix  $A$ . The Cauchy-Riemann equations (4.6) transform to

$$(4.9) \quad \frac{\partial G}{\partial \bar{\zeta}} = B \frac{\partial G}{\partial \zeta},$$

for  $\zeta = H(z)$ , where  $B$  is given by

$$(4.10) \quad \frac{\partial H}{\partial \bar{z}} + \frac{\partial \bar{H}}{\partial \bar{z}}(B \circ H) = A \left[ \frac{\partial H}{\partial z} + \frac{\partial \bar{H}}{\partial z}(B \circ H) \right],$$

or equivalently

$$(4.11) \quad B \circ H = - \left( \frac{\partial \bar{H}}{\partial \bar{z}} - A \frac{\partial \bar{H}}{\partial z} \right)^{-1} \left( \frac{\partial H}{\partial \bar{z}} - A \frac{\partial H}{\partial z} \right).$$

The formal integrability condition (4.7) implies the corresponding formal integrability of the new Cauchy-Riemann equations, i.e.,

$$(4.12) \quad \frac{\partial B_j}{\partial \bar{\zeta}_\ell} + B_j \frac{\partial B_\ell}{\partial \zeta} = \frac{\partial B_\ell}{\partial \bar{\zeta}_j} + B_\ell \frac{\partial B_j}{\partial \zeta}, \quad 1 \leq j, \ell \leq m,$$

where  $B_j$  are the rows of  $B$ ,  $1 \leq j \leq m$ . Furthermore, if  $B$  satisfies (4.11), then the actual integrability, i.e., the existence of a diffeomorphism  $G$  satisfying (4.9), is equivalent to the actual integrability of  $J$ , i.e., the existence of a diffeomorphism  $F$  satisfying (4.6).

A key idea of [M] to guarantee the existence of a diffeomorphism  $G$  satisfying (4.9) is to construct  $H$  in such a fashion that if  $B$  is defined by (4.11) then

$$(4.13) \quad \sum_j \frac{\partial B_j}{\partial \zeta_j} = 0.$$

Equivalently, the task is to construct a diffeomorphism  $H$  on a neighborhood  $U$  of  $p_0 = 0$  in  $\mathbb{C}^m$  such that, if  $B$  is defined by (4.11), then (4.13) holds. It is convenient to dilate the  $z$ -variable, so that  $A(z)$  in (4.11) is replaced by  $A_t(z) = A(tz)$ , and we solve on the unit ball, which we denote  $U$ , for sufficiently small positive  $t$ . Note that if  $A \in C_*^r$  and  $A(0) = 0$ , then  $\|A_t\|_{C_*^r(\bar{U})} \rightarrow 0$  as  $t \rightarrow 0$ . If we relabel  $A_t$  as  $A$ , we want to establish the following variant of Lemma 3.2 of [HT]. To state it, let us set

$$(4.14) \quad \mathcal{A}^r(\eta) = \left\{ A \in C_*^r(\bar{U}) : A(0) = 0, \|A\|_{C^r(\bar{U})} < \eta \right\}.$$

**Proposition 4.1.** *Assume  $r > 1/2$ . Given  $\varepsilon, \delta > 0$ , there exists  $\eta > 0$  such that for any  $A \in \mathcal{A}^r(\eta)$  one can find*

$$(4.15) \quad H \in C_*^{1+r}(\overline{U}),$$

*satisfying*

$$(4.16) \quad H(0) = 0, \quad \|H - \text{id}\|_{C_*^{1+r}(\overline{U})} < \delta,$$

*and such that  $B \in C_*^r(\overline{U})$ , defined by (4.11), satisfies (4.13), and  $\|B\|_{L^\infty(U)} < \varepsilon$ . Furthermore,  $H$  is obtained as a  $C^1$  map*

$$(4.17) \quad \mathcal{A}^r(\eta) \longrightarrow C_*^{1+r}(\overline{U}), \quad A \mapsto H,$$

*Proof.* Let us set

$$(4.18) \quad \Phi(H, A) = E = -\left(\frac{\partial \overline{H}}{\partial \overline{z}} - A \frac{\partial \overline{H}}{\partial z}\right)^{-1} \left(\frac{\partial H}{\partial \overline{z}} - A \frac{\partial H}{\partial z}\right).$$

Then  $\Phi$  is a  $C^1$  map

$$(4.19) \quad \Phi : \mathcal{B}^{r+1}(\delta) \times \mathcal{A}^r(1) \longrightarrow C^r(\overline{U}),$$

where  $\mathcal{A}^r(\eta)$  is as in (4.14) and

$$(4.20) \quad \mathcal{B}^{r+1}(\delta) = \left\{ H \in C_*^{1+r}(\overline{U}) : H(0) = 0, \ \|H - \text{id}\|_{C_*^{1+r}(\overline{U})} < \delta \right\}.$$

If  $B \circ H = \Phi(H, A)$ , an application of the chain rule gives

$$(4.21) \quad \frac{\partial B_j}{\partial \zeta_j} \circ H = \left(\frac{\partial K}{\partial \zeta_j} \circ H\right) \frac{\partial E_j}{\partial z} + \left(\frac{\partial \overline{K}}{\partial \zeta_j} \circ H\right) \frac{\partial E_j}{\partial \overline{z}}, \quad K = H^{-1}.$$

Using the identity

$$(4.22) \quad (DK) \circ H(z) = DH(z)^{-1}$$

of real  $(2m) \times (2m)$  matrices, one can express  $(\partial K / \partial \zeta_j) \circ H$  and  $(\partial \overline{K} / \partial \zeta_j) \circ H$  in terms of the  $z$ - and  $\overline{z}$ -derivatives of  $H$  and  $\overline{H}$ . It follows from Lemma 1.2 of [HT] (extended to function spaces on bounded domains) that

$$(4.23) \quad \Psi(H, A) = \sum_j \frac{\partial B_j}{\partial \zeta_j} \circ H$$

defines a  $C^1$  map

$$(4.24) \quad \Psi : \mathcal{B}^{r+1}(\delta) \times \mathcal{A}^r(1) \longrightarrow C_*^{r-1}(\overline{U}).$$

In fact  $H \mapsto \Psi(H, A)$  is given by a nonlinear second order differential operator:

$$(4.25) \quad \Psi(H, A) = \sum_j a_j(\nabla H) \partial_j b_j(A, \nabla H),$$

where  $a_j$  and  $b_j$  are smooth in their arguments. We note that if

$$(4.26) \quad H(z) = z + \varepsilon h(z),$$

then

$$(4.27) \quad \Phi(H, 0) = -\varepsilon \frac{\partial h}{\partial \bar{z}} + O(\varepsilon^2),$$

and (for  $A = 0$ )

$$(4.28) \quad \frac{\partial B_j}{\partial \zeta_j} \circ H = -\varepsilon \frac{\partial^2 h}{\partial z_j \partial \bar{z}_j} + O(\varepsilon^2).$$

Hence

$$(4.29) \quad \Psi(\text{id}, 0) = 0,$$

and

$$(4.30) \quad D_H \Psi(\text{id}, 0)h = - \sum_j \frac{\partial^2 h}{\partial z_j \partial \bar{z}_j} = -\frac{1}{4} \Delta h.$$

The map (4.30) has a right inverse

$$(4.31) \quad \tilde{G}h = -4(Gh - Gh(0)),$$

where  $G$  denotes the solution operator to

$$(4.32) \quad \Delta v = h \text{ on } U, \quad v|_{\partial U} = 0,$$

which has the mapping property

$$(4.33) \quad G : C_*^{r-1}(\overline{U}) \longrightarrow C_*^{r+1}(\overline{U}),$$



valid for  $r > 0$ . From here, Proposition 4.1 follows from the Implicit Function Theorem.

If  $A \in \mathcal{A}^r(\eta)$  satisfies the formal integrability condition (4.7) and we construct  $H$  according to Proposition 4.1, defining  $B$  by (4.11), then  $B$  satisfies both (4.12) and (4.13). This is an overdetermined elliptic system (if  $\varepsilon$  is small enough), which we will write as

$$(4.34) \quad \sum_{|\alpha|=1} a_\alpha(B) \partial^\alpha B = 0.$$

The a priori regularity we have on  $B$  from (4.11) is

$$(4.35) \quad B \circ H \in C^r(\overline{U}), \quad \text{hence } B \in C^r(\overline{\mathcal{O}}),$$

where  $\overline{\mathcal{O}} \subset H^{-1}(\overline{U})$ . As shown in Lemma 4.1 of [HT], having this a priori information with  $r > 1/2$  allows us to obtain

$$(4.36) \quad B \in C_{\text{loc}}^N(\mathcal{O}),$$

for each  $N < \infty$ . Then classical results yield

$$(4.37) \quad |\partial^\alpha B(\zeta)| \leq C^{|\alpha|+1} \alpha!, \quad \zeta \in \mathcal{O}^b \subset \subset \mathcal{O}, \quad C = C(\mathcal{O}^b).$$

Once we have this (as [M] noted), producing a diffeomorphism  $G$  such that (4.9) holds, which amounts to proving the Newlander-Nirenberg theorem in the real analytic setting, is amenable to classical techniques for solving real analytic systems of partial differential equations. A self contained treatment of a complex Frobenius theorem in the real analytic category, which will produce such a construction, is presented in Appendix A of this paper.

Having described how to obtain the holomorphic coordinate system (4.8), we want to examine how it depends on  $A$ . So we pick

$$(4.38) \quad A_1, A_2 \in \mathcal{A}^r(\eta),$$

with  $r > 1/2$  and  $\eta > 0$  sufficiently small, and turn to the task of estimating, in turn (with obvious notation),  $H_1 - H_2$ ,  $B_1 - B_2$ ,  $G_1 - G_2$ , and then  $F_1 - F_2$ , in terms of  $A_1 - A_2$ . The assertion from Proposition 4.1 that the map (4.17) is  $C^1$  leads immediately to our first estimate:

$$(4.39) \quad \|H_1 - H_2\|_{C_*^{1+r}(\overline{U})} \leq C \|A_1 - A_2\|_{C_*^r(\overline{U})}.$$

We also have a  $C^1$  map

$$(4.40) \quad \mathcal{A}^r(\eta) \rightarrow C_*^r(\overline{U}), \quad A \mapsto \tilde{B} = B \circ H,$$

in light of the formula (4.11). Hence

$$(4.41) \quad \|\tilde{B}_1 - \tilde{B}_2\|_{C_*^r(\overline{U})} \leq C\|A_1 - A_2\|_{C_*^r(\overline{U})}.$$

Now  $B_j = \tilde{B}_j \circ K_j$  with  $K_j = H_j^{-1}$ . While  $A \mapsto H$  is  $C^1$  from  $\mathcal{A}^r(\eta)$  to  $C_*^{r+1}(\overline{U})$ , one has that  $A \mapsto K = H^{-1}$  is a continuous map from  $\mathcal{A}^r(\eta)$  to  $C_*^{r+1}(\overline{\mathcal{O}})$  and a  $C^1$  map to  $C_*^r(\overline{\mathcal{O}})$ , where  $\overline{\mathcal{O}}$  is a neighborhood of 0 containing  $H^{-1}(\overline{U})$  for all  $H$  as in (4.16). Consequently

$$(4.42) \quad \|K_1 - K_2\|_{C_*^r(\overline{\mathcal{O}})} \leq C\|A_1 - A_2\|_{C_*^r(\overline{U})}, \quad \|K_j\|_{C_*^{1+r}(\overline{\mathcal{O}})} \leq C.$$

Let us write

$$(4.43) \quad B_1 - B_2 = \tilde{B}_1 \circ K_1 - \tilde{B}_2 \circ K_1 + \tilde{B}_2 \circ K_1 - \tilde{B}_2 \circ K_2.$$

We have

$$(4.44) \quad \|\tilde{B}_1 \circ K_1 - \tilde{B}_2 \circ K_1\|_{C^r(\overline{\mathcal{O}})} \leq C\|\tilde{B}_1 - \tilde{B}_2\|_{C^r(\overline{U})}\|K_1\|_{C^1(\overline{\mathcal{O}})}, \quad 0 < r \leq 1,$$

and

$$(4.45) \quad \|\tilde{B}_2 \circ K_1 - \tilde{B}_2 \circ K_2\|_{L^\infty(\overline{\mathcal{O}})} \leq \|\tilde{B}_2\|_{C^r(\overline{U})}\|K_1 - K_2\|_{L^\infty(\overline{\mathcal{O}})}^r, \quad 0 < r \leq 1.$$

Putting together (4.43)–(4.45), using the estimates (4.41)–(4.42), we obtain

$$(4.46) \quad \|B_1 - B_2\|_{L^\infty(\overline{\mathcal{O}})} \leq C(\|A_2\|_{C^\rho(\overline{U})})\|A_1 - A_2\|_{C^s(\overline{U})}^\rho, \quad \frac{1}{2} < \rho, s < 1.$$

(It is convenient to replace  $r$  by  $\rho$  in our use of (4.45) and to replace  $r$  by  $s$  in our use of (4.42) and (4.44). Typically we will want to take  $\rho$  as large as possible and  $s$  as small as possible.) The estimate (4.46) is a relatively weak estimate, a consequence of the rather rough dependence of  $\tilde{B} \circ K$  on  $K$ . Fortunately, (4.46) can be improved substantially via use of the fact that  $B_1$  and  $B_2$  both satisfy the elliptic system (4.34). Hence  $V = B_1 - B_2$  solves

$$(4.47) \quad \sum_{|\alpha|=1} a_\alpha(B_1) \partial^\alpha V = \sum_{|\alpha|=1} [a_\alpha(B_2) - a_\alpha(B_1)] \partial^\alpha B_2.$$

In fact, as one sees from (4.12)–(4.13),  $a_\alpha(B) = a_\alpha^0 + M_\alpha B$ , with  $M_\alpha$  a linear map, and hence  $V$  solves the linear elliptic system (with real analytic coefficients)

$$(4.48) \quad \sum_{|\alpha|=1} a_\alpha(B_1) \partial^\alpha V - \sum_{|\alpha|=1} (\partial^\alpha B_2) M_\alpha V = 0.$$

The estimates (4.37) hold for  $B_1$  and  $B_2$ . Local elliptic regularity results yield

$$(4.49) \quad |\partial^\alpha (B_1(\zeta) - B_2(\zeta))| \leq C^{|\alpha|+1} \alpha! \|B_1 - B_2\|_{L^\infty(\overline{\mathcal{O}})}, \quad \zeta \in \mathcal{O}^b \subset \subset \mathcal{O}.$$

Then the method of solving (4.9) covered in Appendix A gives

$$(4.50) \quad |\partial^\alpha (G_1(\zeta) - G_2(\zeta))| \leq C^{|\alpha|+1} \alpha! \|B_1 - B_2\|_{L^\infty(\overline{\mathcal{O}})}, \quad \zeta \in \mathcal{O}^b.$$

Now, with  $F_j = G_j \circ H_j$ , we can set

$$(4.51) \quad F_1 - F_2 = G_1 \circ H_1 - G_2 \circ H_1 + G_2 \circ H_1 - G_2 \circ H_2.$$

Under the bounds on  $H_j$  in  $C^{1+r}$  and on  $G_j$  in  $C^N$  produced above, we have, for  $U^b \subset \subset U$ ,  $r \in (1/2, 1)$ ,

$$(4.52) \quad \begin{aligned} \|G_1 \circ H_1 - G_2 \circ H_1\|_{C^{1+r}(U^b)} &\leq C \|G_1 - G_2\|_{C^{1+r}(\mathcal{O}^b)} \\ &\leq C \|B_1 - B_2\|_{L^\infty(\overline{\mathcal{O}})}, \end{aligned}$$

and

$$(4.53) \quad \begin{aligned} \|G_2 \circ H_1 - G_2 \circ H_2\|_{C^{1+r}(U^b)} &\leq C \|G_2\|_{C^3(\mathcal{O}^b)} \|H_1 - H_2\|_{C^{1+r}(\overline{U})} \\ &\leq C \|A_1 - A_2\|_{C^r(\overline{U})}. \end{aligned}$$

Hence

$$(4.54) \quad \begin{aligned} \|F_1 - F_2\|_{C^{1+r}(U^b)} &\leq C \left( \|B_1 - B_2\|_{L^\infty(\overline{\mathcal{O}})} + \|A_1 - A_2\|_{C^r(\overline{U})} \right) \\ &\leq C (\|A_2\|_{C^\rho(\overline{U})}) \|A_1 - A_2\|_{C^s(\overline{U})}^\rho + C \|A_1 - A_2\|_{C^r(\overline{U})}, \end{aligned}$$

given  $1/2 < r, s, \rho < 1$ . For the last inequality, we have used (4.46). As in that estimate, we typically want to take  $\rho$  as large as possible and  $s$  as small as possible.

## 5. Structure of Levi-flat CR-manifolds

In this section we assume  $\mathcal{S}$  is a Lipschitz subbundle of  $\mathbb{C}T\Omega$ , satisfying

$$(5.1) \quad \mathcal{S}_p \cap \overline{\mathcal{S}}_p = 0, \quad \forall p \in \Omega.$$

Hence  $\mathcal{S}_p + \overline{\mathcal{S}}_p$  has constant dimension (say  $k$ ), and so does  $\mathcal{E}_p$ , defined by (1.6). It follows that  $\mathcal{E}$  and  $\mathcal{S} + \overline{\mathcal{S}}$  are Lipschitz vector bundles, and of course  $\mathcal{V} = 0$ . The bundle  $\mathcal{E} \subset T\Omega$  gets a complex structure

$$(5.2) \quad J \in \text{Lip}(\Omega, \text{End } \mathcal{E}),$$

and

$$(5.3) \quad \mathcal{S}_p = \{u + iJu : u \in \mathcal{E}_p\}.$$

We make the involutivity hypotheses (1.1)–(1.2). As explained in the introduction, this is equivalent to the hypothesis that  $\mathcal{E}$  is involutive plus the hypothesis that the Nijenhuis tensor of  $J$  vanishes. A manifold  $\Omega$  with such a structure  $(\mathcal{E}, J)$  is said to be a Levi-flat CR-manifold.

In this setting, a function  $f$  on an open set  $\mathcal{O} \subset \Omega$  is called a CR function provided

$$(5.4) \quad Zf = 0 \text{ on } \mathcal{O}, \quad \forall Z \in \text{Lip}(\mathcal{O}, \mathcal{S}),$$

or equivalently

$$(5.5) \quad Xf + i(JX)f = 0 \text{ on } \mathcal{O}, \quad \forall X \in \text{Lip}(\mathcal{O}, \mathcal{E}).$$

Given the regularity of  $X$  and  $Z$ , we see that  $Zf$  is a well defined distribution for any  $f \in L^2_{\text{loc}}(\mathcal{O})$ . Our goal here is to construct a rich class of CR functions  $f$  having the regularity

$$(5.6) \quad f, Xf \in C^s(\mathcal{O}), \quad \forall X \in \text{Lip}(\mathcal{O}, \mathcal{E}),$$

given  $s < 1/2$ . In fact  $f$  and  $Xf$  will have further regularity along the leaves of the foliation tangent to  $\mathcal{E}$ , as will be explained below.

To begin the construction of such CR functions, we implement the results of §§2–3. For any  $p \in \Omega$ , there are a neighborhood  $U_1$  of  $p$ , a neighborhood  $U_0$  of  $0 \in \mathbb{R}^n$  ( $n = \dim \Omega$ ) and a bi-Lipschitz map  $\mathcal{G} : U_0 \rightarrow U_1$ , pulling  $\mathcal{E}$  back to the bundle  $\mathcal{E}^\#$  spanned by  $\partial/\partial t_1, \dots, \partial/\partial t_k$ , where in  $U_0 \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$  we have coordinates  $(t, z) = (t_1, \dots, t_k, z_1, \dots, z_{n-k})$ . Furthermore, Lipschitz sections of  $\mathcal{E}$  are transformed to Lipschitz vector fields on  $U_0$ , and  $J$  is transformed to

$$(5.7) \quad J_0 \in \text{Lip}(U_0, \text{End } \mathcal{E}^\#).$$

We may as well assume  $U_0 = U'_0 \times U''_0$ , where  $U'_0$  is a neighborhood of  $0 \in \mathbb{R}^k$  and  $U''_0$  a neighborhood of  $0 \in \mathbb{R}^{n-k}$ . Then  $J_0 = J_0(z)$  is effectively a family of integrable almost complex structures on  $U'_0$ , parametrized by  $z \in U''_0$ . Of course  $k$  is even; say  $k = 2m$ .

Now we can apply the results of §4. We construct holomorphic functions  $F = (f_1, \dots, f_m)$  on  $U'_0$ , depending on  $z$  as a parameter, say  $F = F_z : U'_0 \rightarrow \mathbb{C}^m$ ,  $z \in U''_0$ . (Note that  $z$  has a different role here than in §4; this should not cause confusion.) We construct  $F_z$  as a composition:

$$(5.8) \quad F_z(t) = G_z(H_z(t)).$$

The family of diffeomorphisms  $H_z$  is constructed in Proposition 4.1, via an implicit function theorem. Perhaps after shrinking  $U'_0$  and  $U''_0$ , we have  $H_z \in C^{1+r}(U'_0)$  for each  $z \in U'_0$ , given  $r < 1$ , and

$$(5.9) \quad \begin{aligned} \|H_z - H_{z'}\|_{C^{1+r}(U'_0)} &\leq C\|A_z - A_{z'}\|_{C^2(U'_0)} \\ &\leq C|z - z'|^{1-r}, \end{aligned}$$

if  $1/2 < r < 1$ . Here we have used

$$(5.10) \quad \|A_1 - A_2\|_{C^r} \leq C\|A_1 - A_2\|_{L^\infty}^{1-r} \|A_1 - A_2\|_{\text{Lip}}^r, \quad 0 < r < 1.$$

As explained in §4, the construction of  $G_z$  follows from the real-analytic version of the Newlander-Nirenberg theorem, a presentation of which is given here, in Appendix A. Then we obtain  $F_z = G_z \circ H_z$ , and, by (4.54), with  $U_0^b \subset\subset U'_0$ ,

$$(5.11) \quad \|F_z - F_{z'}\|_{C^{1+r}(U_0^b)} \leq C(\|A_{z'}\|_{C^\rho(U'_0)})\|A_z - A_{z'}\|_{C^s(U'_0)}^\rho + C\|A_z - A_{z'}\|_{C^r(U'_0)},$$

given  $1/2 < r, s, \rho < 1$ . Here we pick  $\rho = 1 - \varepsilon$ ,  $s = 1/2 + \varepsilon$ , and use (5.10) to obtain

$$(5.12) \quad \begin{aligned} \|F_z - F_{z'}\|_{C^{1+r}(U_0^b)} &\leq C|z - z'|^{1/2-\delta} + C|z - z'|^{1-r} \\ &\leq C|z - z'|^{1-r}, \end{aligned}$$

given  $r \in (1/2, 1)$ , and taking  $\varepsilon$  (hence  $\delta$ ) sufficiently small.

The functions  $f_j(t, z)$  given by  $F_z(t) = (f_1(t, z), \dots, f_m(t, z))$  are CR functions on  $U_0$ . In addition, the functions  $\varphi_j(t, z) = z_j$ ,  $1 \leq j \leq n - k$ , are CR functions on  $U_0$ . Then

$$(5.13) \quad \Phi(t, z) = (f_1(t, z), \dots, f_m(t, z), z_1, \dots, z_{n-k})$$

gives a Hölder continuous homeomorphism of  $U_0$  (possibly shrunken some more) onto an open subset of  $\mathbb{C}^m \times \mathbb{R}^{n-k}$ . We compose with  $G^{-1}$  to get associated CR functions on  $U_1 \subset \Omega$ . Let us formally record the result.

**Proposition 5.1.** *Given  $\Omega$  with a Lipschitz, Levi-flat CR structure,  $p \in \Omega$ , there exists a neighborhood  $U_1$  of  $p$  and a homeomorphism*

$$(5.14) \quad \Phi : U_1 \longrightarrow \mathcal{O} \subset \mathbb{C}^m \times \mathbb{R}^{n-k},$$

whose components are CR functions  $\varphi_1, \dots, \varphi_{m+n-k}$  on  $U_1$ . We have

$$(5.15) \quad \varphi_j, X\varphi_j \in C^s(U_1), \quad \forall X \in \text{Lip}(U_1, \mathcal{E}),$$

for any  $s < 1/2$ . Furthermore,  $\Phi$  is a  $C^{1+r}$ -embedding of each leaf in  $U_1$ , tangent to  $\mathcal{E}$ , into  $\mathbb{C}^m \times \mathbb{R}^{n-k}$ , for each  $r < 1$ .

REMARK. Note that if  $\psi$  is a smooth function on a neighborhood of the range of  $\Phi$  in  $\mathbb{C}^m \times \mathbb{R}^{n-k}$  and if  $\psi$  is holomorphic in the  $\mathbb{C}^m$ -variables, then  $\psi(\varphi_1, \dots, \varphi_{m+n-k})$  is a CR function on  $U_1$ .

If  $\dim \mathcal{S}_p = 1$ , so  $k = 2$  and the leaves tangent to  $\mathcal{E}$  are 2-dimensional, then we can use the results of Appendix B in place of those of §4. Consequently we can improve the regularity result (5.15) to

$$(5.16) \quad |\varphi_j(x) - \varphi_j(x')|, |X\varphi_j(x) - X\varphi_j(x')| \leq C\sigma^\#(|x - x'|),$$

where, given  $a > 0$ ,

$$(5.17) \quad \sigma^\#(\delta) = \delta \left( \log \frac{e}{\delta} \right)^{1+a},$$

for  $0 < \delta \leq 1$ .

We now give a sufficient condition for the existence of a CR embedding  $\Phi$  as in (5.14) that is a  $C^1$  diffeomorphism.

**Proposition 5.2.** *Assume  $\Omega$  is a Levi-flat CR manifold with a CR structure regular of class  $C^\rho$ , with  $\rho > 3/2$ . Then the map  $\Phi$  in (5.14) can be taken to be a  $C^1$  diffeomorphism.*

*Proof.* The new regularity hypothesis is that  $\mathcal{S}$  is a  $C^\rho$  bundle. Thus  $\mathcal{E}$  and  $J$  are regular of class  $C^\rho$ , and these structures pull back to  $C^\rho$  structures under the map  $\mathcal{G}$ , which is a  $C^\rho$  diffeomorphism. In particular,  $A(t, z)$  is a  $C^1$  function of  $z$  with values in  $C^s(U'_0)$ , with  $s = \rho - 1 > 1/2$ . Thus the implicit function theorem argument of Proposition 4.1 yields  $H_z$ , a  $C^1$  function of  $z$  with values in  $C^{1+s}$ . From here, one obtains  $C^1$  dependence of  $G_z$  on  $z$  and the result follows.

Note that if the leaves tangent to  $\mathcal{E}$  are 2-dimensional, we can obtain the conclusion of Proposition 5.2 whenever  $\rho > 1$ , again using the results of Appendix B in place of those of §4.

REMARKS ON THE EMBEDDED CASE. Suppose  $\Omega \subset \mathbb{C}^N$  is a  $C^{1,1}$  submanifold, of real dimension  $d$ , and that  $T_p\Omega \cap JT_p\Omega = \mathcal{E}_p$  has constant real dimension  $k = 2m$ , so  $\Omega$  has the structure of a CR-manifold. The vector bundle  $\mathcal{E} \subset T\Omega$  is a Lipschitz vector bundle, and the condition that  $\mathcal{E}$  be involutive is equivalent to the condition that  $\Omega$  is a Levi-flat CR-manifold. In such a case, the results of §2 imply that  $\Omega$  is foliated by manifolds, of real dimension  $k$ , tangent to  $\mathcal{E}$ , and smooth of class  $C^{1,1}$ .

In this case one does not need the Newlander-Nirenberg theorem (or a refinement) to establish that these leaves are complex manifolds. Rather methods going back to Levi-Civita [LC], and developed further in [Som], [Fr], and [Pin] suffice. Levi-Civita's result for a single leaf is:

**Proposition 5.3.** *If  $M$  is a  $C^1$  submanifold of  $\mathbb{C}^N$  and each tangent space  $T_p M$  is  $J$ -invariant, then  $M$  is a complex manifold.*

*Proof.* Fix  $p \in M$ , and represent  $M$  near  $p$  as the graph over the complex vector space  $V = T_p M$ ; so one has a  $C^1$  diffeomorphism  $G : \mathcal{O} \rightarrow M$  where  $\mathcal{O}$  is a neighborhood of  $0 \in V$ . It is readily verified that  $DG(q)$  is  $\mathbb{C}$ -linear for  $q \in \mathcal{O}$ , so  $G : \mathcal{O} \rightarrow \mathbb{C}^N$  is holomorphic.

In the setting above, we have a family  $M_z$  of leaves, depending in a Lipschitz fashion of  $z \in \mathcal{U} \subset \mathbb{R}^\ell$ , where  $d = \ell + k$ . Given  $p \in \Omega$ , say  $p \in M_{z_0}$ , pick  $V = T_p M_{z_0}$ , and for  $z$  close to  $z_0$  we have  $M_z$  locally a graph over  $\mathcal{O} \subset V$ . The comments above give local holomorphic diffeomorphisms  $G_z : \mathcal{O} \rightarrow M_z \subset \mathbb{C}^N$ . This construction, as we have said, is essentially classical. The one point to make here is that we have the Frobenius theory of [Ha], so we are able to treat submanifolds of class  $C^{1,1}$  while previous treatments take  $\Omega$  to be of class  $C^2$ . In connection with this, we note that Theorem 2.1 of [Pin] refers to CR-manifolds in  $\mathbb{C}^N$  of class  $C^m$ , with  $m \geq 1$ , but a perusal of the proof shows that the author means to say the relevant tangent spaces are smooth of class  $C^m$ , which holds if  $\Omega \subset \mathbb{C}^N$  is a submanifold of class  $C^{m+1}$  (satisfying the CR property).

## 6. The complex Frobenius theorem

We recall our set-up. We have a Lipschitz bundle  $\mathcal{S} \subset \mathbb{C}T\Omega$ , we assume  $\mathcal{S} + \overline{\mathcal{S}}$  is also a Lipschitz bundle, and we assume that

$$(6.1) \quad X, Y \in \text{Lip}(\Omega, \mathcal{S}) \Rightarrow [X, Y] \in L^\infty(\Omega, \mathcal{S}), \quad [X, \overline{Y}] \in L^\infty(\mathcal{S} + \overline{\mathcal{S}}).$$

We then form the Lipschitz bundles  $\mathcal{V} \subset \mathcal{E} \subset T\Omega$ , with fibers

$$(6.2) \quad \mathcal{V}_p = \mathcal{S}_p \cap T_p \Omega, \quad \mathcal{E}_p = \{w + \overline{w} : w \in \mathcal{S}_p\},$$

which therefore satisfy

$$(6.3) \quad \begin{aligned} X, Y \in \text{Lip}(\Omega, \mathcal{E}) &\implies [X, Y] \in L^\infty(\Omega, \mathcal{E}), \\ X, Y \in \text{Lip}(\Omega, \mathcal{V}) &\implies [X, Y] \in L^\infty(\Omega, \mathcal{V}). \end{aligned}$$

Furthermore, we have a complex structure on  $\mathcal{E}/\mathcal{V}$ ,

$$(6.4) \quad J \in \text{Lip}(\Omega, \text{End } \mathcal{E}/\mathcal{V}),$$

satisfying

$$(6.5) \quad J(u \bmod \mathcal{V}) = v \bmod \mathcal{V}, \quad u + iv \in \mathcal{S}_p.$$

Our proximate goal is to construct a Levi-flat CR manifold as a quotient (locally) of  $\Omega$ , via the action of a local group of flows generated by sections of  $\mathcal{V}$ . In order to achieve this, we need a further hypothesis on the regularity with which  $\mathcal{V}$  sits in  $\mathcal{E}$ . One way to put it is the following. Say  $\dim \mathcal{V}_p = \ell \leq k = \dim \mathcal{E}_p$ .

**Hypothesis V.** *Each  $p \in \Omega$  has a neighborhood  $U_1$  on which there is a local Lipschitz frame field  $\{X_1, \dots, X_k\}$  for  $\mathcal{E}$ , such that  $\{X_1, \dots, X_\ell\}$  is a local frame field for  $\mathcal{V}$  and*

$$(6.6) \quad [X_i, X_j] = 0, \quad 1 \leq i, j \leq k.$$

Later we will give other conditions that imply Hypothesis V, but for now we show how it leads to the desired quotient space.

With respect to such a local frame field, for  $x \in U_1$  we can identify  $\mathcal{E}_x/\mathcal{V}_x$  with the linear span of  $X_{\ell+1}(x), \dots, X_k(x)$ , and we can represent  $J$  by a  $(k-\ell) \times (k-\ell)$  matrix:

$$(6.7) \quad JX_j = \sum_{m=\ell+1}^k \mathcal{J}_{jm}(x) X_m \bmod \mathcal{V}_x, \quad \ell+1 \leq j \leq k.$$

Note that if  $Y_j \in \text{Lip}(U_1, \mathcal{E})$  and  $X_j + iY_j \in \text{Lip}(U_1, \mathcal{S})$ , so  $Y_j = JX_j \bmod \mathcal{V}$ , we have

$$(6.8) \quad \begin{aligned} [X_i, X_j + iY_j] &= i[X_i, Y_j] \in L^\infty(\Omega, \mathcal{S}) \cap iL^\infty(\Omega, \mathcal{E}) \\ &\subset iL^\infty(\Omega, \mathcal{V}), \end{aligned}$$

for  $1 \leq i \leq \ell$ ,  $\ell+1 \leq j \leq k$ , by (6.1) and (6.6). Taking  $Y_j$  to be the sum in (6.7), and noting that

$$(6.9) \quad \left[ X_i, \sum_{m=\ell+1}^k \mathcal{J}_{jm} X_m \right] = \sum_{m=\ell+1}^k (X_i \mathcal{J}_{jm}) X_m,$$

again by (6.6), we deduce that (6.9) actually vanishes, and hence

$$(6.10) \quad X_i \mathcal{J}_{jm} = 0, \quad 1 \leq i \leq \ell, \ell+1 \leq j, m \leq k.$$

In a fashion parallel to (2.17) and (3.1), we set

$$(6.11) \quad G(t, z) = \mathcal{F}^t(0, z), \quad \mathcal{F}^t = \mathcal{F}_{X_1}^{t_1} \circ \dots \circ \mathcal{F}_{X_k}^{t_k},$$

with  $X_1, \dots, X_k$  as in Hypothesis V. By Proposition 2.3,  $G : U_0 \rightarrow U_1$  is a bi-Lipschitz map from a neighborhood  $U_0$  of  $(0, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$  to a neighborhood  $U_1$  of  $p \in \Omega$ . We denote by  $\mathcal{V}_0 \subset TU_0$  the pull back of  $\mathcal{V}$ , by  $\mathcal{E}_0 \subset TU_0$  the pull back of  $\mathcal{E}$ , and by  $\mathcal{S}_0 \subset \mathbb{C}TU_0$  the pull back of  $\mathcal{S}$ . Note that  $\mathcal{V}_0$  is spanned by  $\partial/\partial t_j$ ,  $1 \leq j \leq \ell$  and  $\mathcal{E}_0$  by  $\partial/\partial t_j$ ,  $1 \leq j \leq k$ . The quotient bundle  $\mathcal{E}_0/\mathcal{V}_0$  is



isomorphic to the span of  $\partial/\partial t_j$  for  $\ell + 1 \leq j \leq k$ , and the complex structure  $J$  on  $\mathcal{E}/\mathcal{V}$  pulls back to  $J^0$ , given by

$$(6.12) \quad J^0 \frac{\partial}{\partial t_j} = \sum_{m=\ell+1}^k \mathcal{J}_{jm}^0(t, z) \frac{\partial}{\partial t_m}, \quad \mathcal{J}_{jm}^0(t, z) = \mathcal{J}_{jm}(G(t, z)), \quad \ell+1 \leq j \leq k.$$

The result (6.10) is equivalent to

$$(6.13) \quad \frac{\partial}{\partial t_i} \mathcal{J}_{jm}^0(t, z) = 0, \quad 1 \leq i \leq \ell, \ell+1 \leq j, m \leq k,$$

so we can write

$$(6.14) \quad \mathcal{J}_{jm}^0 = \mathcal{J}_{jm}^0(t'', z), \quad t'' = (t_{\ell+1}, \dots, t_k).$$

At this point it is natural to form the quotient space  $\tilde{U}_0 = U_0 / \sim$ , where we use the equivalence relation

$$(6.15) \quad (t, z) \sim (s, z) \iff (t_{\ell+1}, \dots, t_k) = (s_{\ell+1}, \dots, s_k).$$

In other words,  $\tilde{U}_0$  is a neighborhood of  $(0, 0) \in \mathbb{R}^{k-\ell} \times \mathbb{R}^{n-k}$ , with coordinates

$$(6.16) \quad (t_{\ell+1}, \dots, t_k, z_1, \dots, z_{n-k}).$$

Note that  $U_1$  fibers over  $\tilde{U}_0$ , via

$$(6.17) \quad \pi = P \circ G^{-1} : U_1 \longrightarrow \tilde{U}_0,$$

where  $P(t_1, \dots, t_k) = (t_{\ell+1}, \dots, t_k)$ . We will display a Levi-flat CR structure on  $\tilde{U}_0$ , with  $\tilde{\mathcal{E}}_0$  the span of  $\partial/\partial t_j$ ,  $\ell+1 \leq j \leq k$  and

$$(6.18) \quad \tilde{J}^0 \frac{\partial}{\partial t_j} = \sum_{m=\ell+1}^k \mathcal{J}_{jm}^0(t'', z) \frac{\partial}{\partial t_m}.$$

To see this, note that a vector field of the form

$$(6.19) \quad \frac{\partial}{\partial t_j} + i\tilde{J}^0 \frac{\partial}{\partial t_j}, \quad \ell+1 \leq j \leq k,$$

can be regarded as a vector field on either  $\tilde{U}_0$  or  $U_0$ . In the latter guise it is a Lipschitz section of  $\mathcal{S}_0$ . The involutivity condition (6.1) has a counterpart for  $\mathcal{S}_0$ , which implies that the Nijenhuis tensor of  $\tilde{J}^0$  vanishes, so  $\tilde{U}_0$  has a Levi-flat CR structure, associated with  $\tilde{\mathcal{S}}_0$ , the span of vectors of the form (6.19). This establishes the main result of this section, which we state formally.

**Proposition 6.1.** *Assume  $\mathcal{S}$  and  $\mathcal{S} + \bar{\mathcal{S}}$  are Lipschitz subbundles of  $\mathbb{C}T\Omega$ , satisfying the involutivity condition (6.1) and also Hypothesis V. Then each  $p \in \Omega$  has a neighborhood  $U_1$  and a Lipschitz fibration  $\pi : U_1 \rightarrow \tilde{U}_0$  onto a Levi-flat CR manifold, associated to a Lipschitz subbundle  $\tilde{\mathcal{S}}_0 \subset \mathbb{C}T\tilde{U}_0$ , such that*

$$(6.20) \quad \mathcal{S}|_{U_1} = (D\pi)^{-1} \tilde{\mathcal{S}}_0|_{\tilde{U}_0}.$$

We show that additional regularity conditions on  $\mathcal{V}$  and  $\mathcal{E}$  imply Hypothesis V.

**Proposition 6.2.** *Assume each  $p \in \Omega$  has a neighborhood on which there is a frame field  $\{W_1, \dots, W_k\}$  for  $\mathcal{E}$ , of class  $C^{1,1}$ , such that  $\{W_1, \dots, W_\ell\}$  is a local frame field for  $\mathcal{V}$ . Then Hypothesis V holds.*

*Proof.* We begin with a construction parallel to (2.2)–(2.6), obtaining a local  $C^{1,1}$  frame field  $\{Y_1, \dots, Y_k\}$  for  $\mathcal{E}$  such that  $[Y_i, Y_j] = 0$  for  $1 \leq i, j \leq k$  (though  $\{Y_1, \dots, Y_\ell\}$  might not be a local frame field for  $\mathcal{V}$ ). As in (2.17)–(2.18), construct a diffeomorphism  $G$ , of class  $C^{1,1}$ , via which  $Y_j$  are transformed to  $\partial/\partial t_j$ ,  $1 \leq j \leq k$ , and note that  $W_j$  are transformed to  $C^{1,1}$  vector fields  $V_j = \sum_{i=1}^k v_{ji}(t, z) \partial/\partial t_i$ . Now produce a  $C^{1,1}$  diffeomorphism  $H$  that straightens appropriate linear combinations of  $V_1, \dots, V_\ell$  to  $\partial/\partial s_1, \dots, \partial/\partial s_\ell$ , while each  $\partial/\partial s_j$  ( $1 \leq j \leq k$ ) is a linear combination of  $\partial/\partial t_1, \dots, \partial/\partial t_k$ . Then transform  $\partial/\partial s_j$  via  $(H \circ G)^{-1}$ , to obtain the Lipschitz vector fields  $X_j$  of Hypothesis V.

## A. A Frobenius theorem for real analytic, complex vector fields

Let  $X_1, \dots, X_m$  be real analytic, complex vector fields on an open set  $\mathcal{O} \subset \mathbb{R}^n$ . We assume

$$(A.1) \quad [X_k, X_\ell] = 0, \quad 1 \leq k, \ell \leq m.$$

We want to obtain conditions under which we can find real analytic solutions  $u$  to

$$(A.2) \quad X_k u = 0, \quad 1 \leq k \leq m,$$

on a neighborhood of a given point  $p \in \mathcal{O}$ . We proceed as follows. Say

$$(A.3) \quad X_k = \sum_j a_{kj}(x) \frac{\partial}{\partial x_j}.$$

On a neighborhood  $\Omega$  of  $p$  in  $\mathbb{C}^n$  set

$$(A.4) \quad Z_k = \sum_j a_{kj}(z) \frac{\partial}{\partial z_j},$$

with  $a_{kj}(z)$  holomorphic extensions of  $a_{kj}(x)$ . Solving (A.2) is equivalent to finding a holomorphic solution  $u$  to

$$(A.5) \quad Z_k u = 0, \quad 1 \leq k \leq m,$$

on a neighborhood of  $p$  in  $\mathbb{C}^n$ . Note that (A.1) implies

$$(A.6) \quad [Z_k, Z_\ell] = 0, \quad 1 \leq k, \ell \leq m.$$

Our next step involves passing to real vector fields on  $\Omega \subset \mathbb{C}^n \approx \mathbb{R}^{2n}$ . Generally, if

$$(A.7) \quad Z = \sum_j a_j(z) \frac{\partial}{\partial z_j},$$

set

$$(A.8) \quad a_j(z) = f_j(z) + i g_j(z),$$

with  $f_j$  and  $g_j$  real valued, and then set

$$(A.9) \quad \Phi(Z) = Y = \sum_j \left( f_j \frac{\partial}{\partial x_j} + g_j \frac{\partial}{\partial y_j} \right).$$

If  $Z$  is a holomorphic vector field, i.e., if (A.7) holds with  $a_j(z)$  holomorphic, we say  $Y = \Phi Z$  is a *real-holomorphic vector field*. Our first lemma holds whether or not the coefficients of  $Z$  are holomorphic.

**Lemma A.1.** *If  $a_j \in C(\Omega)$  in (A.6) and  $Y = \Phi(Z)$ , then*

$$(A.10) \quad u \text{ holomorphic} \implies Zu = Yu.$$

The proof is a straightforward calculation, making use of

$$(A.11) \quad \frac{\partial u}{\partial z_j} = \frac{\partial u}{\partial x_j} = \frac{1}{i} \frac{\partial u}{\partial y_j}.$$

The following is special to holomorphic vector fields, namely that  $\Phi$  preserves the Lie bracket when applied to such vector fields.

**Lemma A.2.** *If also  $W = \sum b_j(z) \partial/\partial z_j$ , then*

$$(A.12) \quad a_j, b_j \text{ holomorphic} \implies \Phi[Z, W] = [\Phi Z, \Phi W].$$

Again the proof is a straightforward (though slightly tedious) calculation.

It follows that if  $X_k$  and  $Z_k$  are as in (A.3)–(A.4), and if

$$(A.13) \quad Y_k = \Phi Z_k = \sum_j \left( f_{kj} \frac{\partial}{\partial x_j} + g_{kj} \frac{\partial}{\partial y_j} \right), \quad f_{kj} = \operatorname{Re} a_{kj}, \quad g_{kj} = \operatorname{Im} a_{kj},$$

then

$$(A.14) \quad [X_k, X_\ell] = 0 \Rightarrow [Z_k, Z_\ell] = 0 \Rightarrow [Y_k, Y_\ell] = 0, \quad 1 \leq k, \ell \leq m.$$

The complex structure on  $\mathbb{C}^n$  produces a complex structure on the space of real vector fields on  $\Omega$ , defined by

$$(A.15) \quad J \frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}, \quad J \frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}.$$

Note that if  $Z$  has the form (A.7), then

$$(A.16) \quad \Phi(iZ) = J \Phi(Z).$$

In particular, if  $Y_k$  are as in (A.13),

$$(A.17) \quad [Y_k, JY_\ell] = 0 = [JY_k, JY_\ell], \quad 1 \leq k, \ell \leq m.$$

One advantage of using the real vector fields  $Y_k$  on  $\Omega$  is that they generate local flows  $\mathcal{F}_{Y_k}^t$  on  $\Omega$ . In this context, the following results are very useful.

Suppose  $Y$  is a real-holomorphic vector field on  $\Omega$ . It follows from (A.16) that so is  $JY$ , and  $Y$  and  $JY$  commute. Thus so do the local flows  $\mathcal{F}_Y^s$  and  $\mathcal{F}_{JY}^t$ . The following gives important information on how these flows fit together.

**Proposition A.3.** *If  $Y$  is a real-holomorphic vector field on  $\Omega$ , then, for each  $z \in \Omega$ ,*

$$(A.18) \quad \mathcal{F}_Y^s \mathcal{F}_{JY}^t(z) \text{ is holomorphic in } s + it.$$

*Proof.* Denote the 2-parameter orbit in (A.18) by  $\varphi(s, t)$ . By commutativity we also have

$$(A.19) \quad \varphi(s, t) = \mathcal{F}_{JY}^t \mathcal{F}_Y^s(z).$$

It follows that

$$(A.20) \quad \frac{\partial \varphi}{\partial s} = Y(\varphi(s, t)), \quad \frac{\partial \varphi}{\partial t} = JY(\varphi(s, t)),$$

and hence  $\partial \varphi / \partial t = J \partial \varphi / \partial s$ , which gives the asserted holomorphicity.

The following is an important complement.

**Proposition A.4.** *If  $Y$  is a real-holomorphic vector field on  $\Omega$ , then  $\mathcal{F}_Y^t$  is a local group of holomorphic maps.*

*Proof.* The claim is equivalent to the assertion that

$$(A.21) \quad \mathcal{F}_{Y\#}^t \circ J = J \circ \mathcal{F}_{Y\#}^t,$$

where, given a diffeomorphism  $F$ ,  $F_\#$  is the induced operator on vector fields. One has

$$(A.22) \quad \left. \frac{d}{dt} \mathcal{F}_{Y\#}^t W \right|_{t=0} = [Y, W];$$

cf. (8.3) in Chapter I of [T]. Hence

$$(A.23) \quad \left. \frac{d}{dt} (\mathcal{F}_{Y\#}^t \circ J - J \circ \mathcal{F}_{Y\#}^t) W \right|_{t=0} = [Y, JW] - J[Y, W].$$

If  $Y = \Phi Z$  with  $Z$  a holomorphic vector field, as in (A.7)–(A.9), then a calculation using

$$(A.24) \quad \frac{\partial f_j}{\partial x_\ell} = \frac{\partial g_j}{\partial y_\ell}, \quad \frac{\partial f_j}{\partial y_\ell} = -\frac{\partial g_j}{\partial x_\ell},$$

shows that, for any vector field  $W$ ,

$$(A.25) \quad [Y, JW] - J[Y, W] = 0,$$

so the quantity (A.23) vanishes. More generally,

$$(A.26) \quad \begin{aligned} \frac{d}{dt} (\mathcal{F}_{Y\#}^t \circ J - J \circ \mathcal{F}_{Y\#}^t) W &= \mathcal{F}_{Y\#}^t [Y, JW] - J \mathcal{F}_{Y\#}^t [Y, W] \\ &= (\mathcal{F}_{Y\#}^t \circ J - J \circ \mathcal{F}_{Y\#}^t) [Y, W], \end{aligned}$$

the latter identity by (A.25). An iteration gives

$$(A.27) \quad \left( \frac{d}{dt} \right)^\ell (\mathcal{F}_{Y\#}^t \circ J - J \circ \mathcal{F}_{Y\#}^t) W = (\mathcal{F}_{Y\#}^t \circ J - J \circ \mathcal{F}_{Y\#}^t) (\mathcal{L}_Y^\ell W),$$

where  $\mathcal{L}_Y W = [Y, W]$ . In particular, for all  $\ell \in \mathbb{Z}^+$ ,

$$(A.28) \quad \left( \frac{d}{dt} \right)^\ell (\mathcal{F}_{Y\#}^t \circ J - J \circ \mathcal{F}_{Y\#}^t) W \Big|_{t=0} = 0.$$

In the current context,  $\mathcal{F}_Y^t$  and all its derived quantities are real analytic in  $t$  (as a consequence of Proposition A.3), so (A.21) follows from (A.28).

We proceed to find solutions to (A.2), under appropriate hypotheses. For notational simplicity, assume  $p$  is the origin;  $p = 0 \in \mathbb{R}^n \subset \mathbb{C}^n$ . Suppose

$$(A.29) \quad V \text{ is a linear subspace of } \mathbb{R}^n, \text{ of dimension } n - m,$$

and let

$$(A.30) \quad \tilde{V} \text{ be the complexification of } V,$$

so  $\tilde{V}$  is a complex subspace of  $\mathbb{C}^n$ , of complex dimension  $n - m$  (hence real dimension  $2n - 2m$ ). Let  $v$  be a real analytic function on a neighborhood  $\mathcal{U}$  of 0 in  $V$ , extended to a holomorphic function on a neighborhood  $\tilde{\mathcal{U}}$  of 0 in  $\tilde{V}$ . We assume

$$(A.31) \quad \{Y_k, JY_k : 1 \leq k \leq m\} \text{ is transverse to } \tilde{V},$$

on  $\tilde{\mathcal{U}}$ . In particular,

$$(A.32) \quad \mathbb{C}^n = \mathbb{R}\text{-linear span of } \tilde{V} \text{ and } \{Y_k(0), JY_k(0) : 1 \leq k \leq m\}.$$

Conversely, if (A.32) holds, then (A.31) holds, possibly with  $\tilde{\mathcal{U}}$  shrunken. In such a case, we can set

$$(A.33) \quad u(\zeta) = v(z), \text{ for } z \in \tilde{\mathcal{U}}, \quad \zeta = \mathcal{F}_{Y_1}^{s_1} \mathcal{F}_{JY_1}^{t_1} \cdots \mathcal{F}_{Y_m}^{s_m} \mathcal{F}_{JY_m}^{t_m}(z),$$

and see that  $u$  is holomorphic on a neighborhood of 0 in  $\mathbb{C}^n$  and solves

$$(A.34) \quad Y_k u = JY_k u = 0, \quad 1 \leq k \leq m.$$

Hence, by Lemma A.1, (A.5) holds, hence, possibly shrinking  $\mathcal{U}$ , we have

$$(A.35) \quad X_k u = 0, \quad 1 \leq k \leq m, \quad u|_{\mathcal{U}} = v.$$

A classic example to which this construction applies arises in the real analytic case of the Newlander-Nirenberg theorem. In this setting, one has  $n = 2m$  and takes  $\xi_j = x_j + ix_{j+m}$ ,  $1 \leq j \leq m$ , and

$$(A.36) \quad X_k = \frac{\partial}{\partial \bar{\xi}_k} - \sum_{\ell=1}^m b_{k\ell}(x) \frac{\partial}{\partial \xi_\ell}, \quad 1 \leq k \leq m, \quad b_{k\ell}(0) = 0.$$

These vector fields arise from an almost complex structure  $J_0$  on  $\mathcal{O} \subset \mathbb{R}^n$ , and the integrability condition is that they commute, i.e., that (A.1) holds. Then a function  $u$  on  $\mathcal{O}$  is holomorphic with respect to this almost complex structure if and only if

(A.2) holds, and the theorem is that if (A.1) holds then there are  $m$  such functions forming a local coordinate system, in a neighborhood of 0. In this case we have

$$(A.37) \quad X_k(0) = \frac{\partial}{\partial \bar{\xi}_k} = \frac{1}{2} \frac{\partial}{\partial x_k} + \frac{i}{2} \frac{\partial}{\partial x_{m+k}}, \quad 1 \leq k \leq m,$$

hence

$$(A.38) \quad Z_k(0) = \frac{1}{2} \frac{\partial}{\partial z_k} + \frac{i}{2} \frac{\partial}{\partial z_{m+k}},$$

so

$$(A.39) \quad Y_k(0) = \frac{1}{2} \frac{\partial}{\partial x_k} + \frac{1}{2} \frac{\partial}{\partial y_{m+k}},$$

and

$$(A.40) \quad JY_k(0) = \frac{1}{2} \frac{\partial}{\partial y_k} - \frac{1}{2} \frac{\partial}{\partial x_{m+k}}.$$

Let us take for  $V \subset \mathbb{R}^n$  the space

$$(A.41) \quad V = \{x \in \mathbb{R}^n : x_{m+1} = \cdots = x_{2m} = 0\},$$

so

$$(A.42) \quad \tilde{V} = \{x + iy \in \mathbb{C}^n : x_{m+1} = \cdots = x_{2m} = y_{m+1} = \cdots = y_{2m} = 0\},$$

which is spanned over  $\mathbb{R}$  by

$$(A.43) \quad \left\{ \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} : 1 \leq j \leq m \right\}.$$

It is clear that if  $Y_k(0)$  and  $JY_k(0)$  are given by (A.39)–(A.40), then (A.32) holds, so we have solutions to (A.35) in this case, for some neighborhood  $\mathcal{U}$  of 0 in  $V$ , and arbitrary real analytic  $v$  on  $\mathcal{U}$ . This provides enough  $J_0$ -holomorphic functions on a neighborhood of 0 in  $\mathbb{R}^n$  to yield a coordinate system. In this fashion the real analytic case of the Newlander-Nirenberg theorem is proven.

## B. The case of two-dimensional leaves

Here we put ourselves in the setting of §3, and take the Lipschitz bundle  $\mathcal{E}$  to have fiber dimension  $k = 2m = 2$ . We assume  $\mathcal{E}$  has a complex structure  $J$ ,

pulled back as in §3 to a complex structure  $J^0 \in \text{End}(\mathcal{E}_0)$ , where  $\mathcal{E}_0 \subset TU_0$  is the bundle spanned by  $\partial/\partial t_1, \partial/\partial t_2$ . Here  $U_0 \subset \mathbb{R}^n$  is an open set with coordinates  $(t, z)$ ,  $t \in \mathbb{R}^2$ ,  $z \in \mathbb{R}^{n-2}$ . We assume

$$(B.1) \quad J \in C^r(U_1),$$

with  $r \in (0, 1)$ , in which case

$$(B.2) \quad J^0 \in C^r(U_0).$$

We can represent  $J^0 = J^0(t, z)$  as a  $2 \times 2$  matrix valued function of  $(t, z)$ . Making a preliminary change of coordinates

$$(B.3) \quad (t, z) \mapsto (A(z)t, z),$$

where  $A(z)$  is a  $GL(2, \mathbb{R})$ -valued function of the same type of regularity as  $J^0$  in (B.2), we can arrange that

$$(B.4) \quad J^0(0, z) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

for all  $z$ .

In order to implement the classical method of finding isothermal coordinates, we impose a family of Riemannian metric tensors on  $t$ -space, depending on  $z$  as a parameter,  $(g_{ij}(t, z))$ ,  $1 \leq i, j \leq 2$ . Arrange that  $J^0(t, z)$  is an isometry on  $T_t \mathbb{R}^2$  with respect to the induced inner product, for each  $(t, z)$ . One could, for example, start with the standard flat metric  $(\delta_{ij})$  and average with respect to the  $\mathbb{Z}/(4)$ -action generated by  $J^0$ . We then obtain

$$(B.5) \quad g_{ij} \in C^r(U_0),$$

when (B.2) holds, and we can arrange that

$$(B.6) \quad g_{ij}(0, z) = \delta_{ij}.$$

Let  $D = \{t \in \mathbb{R}^2 : t_1^2 + t_2^2 < 1\}$ . We want to find a harmonic function  $u_1$  on  $D$  equal to  $t_1$  on  $\partial D$  (and depending on the parameter  $z$ ). Thus, with  $a^{ij}(t, z) = g(t, z)^{1/2} g^{ij}(t, z)$ , where  $(g^{ij})$  is the inverse of the matrix  $(g_{ij})$  and  $g$  its determinant, we want to solve

$$(B.7) \quad \partial_i a^{ij}(t, z) \partial_j u_1 = 0 \quad \text{on } D, \quad u_1|_{\partial D} = t_1,$$

where  $\partial_i = \partial/\partial t_i$ ,  $i = 1, 2$ . Without changing notation, we dilate the  $t$ -coordinates, and we can assume

$$(B.8) \quad a^{ij}(0, z) = \delta^{ij}, \quad \|a^{ij}(\cdot, z) - \delta^{ij}\|_{C^r(\overline{D})} \leq \eta,$$



where  $\eta > 0$  is a sufficiently small quantity. Let us write (B.7) as

$$(B.9) \quad (\Delta + R_z)u_1 = 0, \quad u_1|_{\partial D} = t_1,$$

where

$$(B.10) \quad R_z u_1 = \partial_i r^{ij}(t, z) \partial_j u_1, \quad r^{ij}(t, z) = a^{ij}(t, z) - \delta^{ij},$$

and

$$\Delta = \frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2}.$$

To establish solvability of (B.9), when  $\eta$  in (B.8) is small enough, note that it is equivalent to the following equation for  $v = u_1 - t_1$ :

$$(B.11) \quad (\Delta + R_z)v = -R_z t_1, \quad v|_{\partial D} = 0,$$

hence to the equation

$$(B.12) \quad (I + GR_z)v = -GR_z t_1,$$

where  $G$  is the solution operator to the Poisson problem for  $\Delta$  on  $D$ , with the Dirichlet boundary condition. Such  $G$  has the property

$$(B.13) \quad G : C_*^{r-1}(\overline{D}) \longrightarrow C^{r+1}(\overline{D}), \quad 0 < r < 1;$$

cf. [T], Chapter 13, (8.54)–(8.55). Hence

$$(B.14) \quad \begin{aligned} \|GR_z f\|_{C^{r+1}(\overline{D})} &\leq C \|r^{ij}(\cdot, z) \partial_j f\|_{C^r(\overline{D})} \\ &\leq C \|r^{ij}(\cdot, z)\|_{C^r(\overline{D})} \|f\|_{C^{r+1}(\overline{D})}, \end{aligned}$$

so if  $\eta$  is small enough, the operator norm of  $GR_z$  on  $C^{r+1}(\overline{D})$  is  $\leq 1/2$ , so  $I + GR_z$  in (B.12) is invertible on  $C^{r+1}(\overline{D})$ , and we have a unique solution  $v$ , satisfying

$$(B.15) \quad \begin{aligned} \|v\|_{C^{r+1}(\overline{D})} &\leq C \|GR_z t_1\|_{C^{r+1}(\overline{D})} \\ &\leq C \|r^{i1}(\cdot, z)\|_{C^r(\overline{D})} \\ &\leq C\eta. \end{aligned}$$

We now have  $u_1 = t_1 + v$ . The standard construction of the harmonic conjugate  $u_2$ , satisfying

$$(B.16) \quad du_2 = (J^0)^t du_1, \quad u_2(0, z) = 0,$$

gives

$$(B.17) \quad \|u_2 - t_2\|_{C^{r+1}(\overline{D})} \leq C\eta,$$

and taking  $u = u_1 + iu_2$ , we have a local holomorphic coordinate system on each leaf  $z = z_0$ , if  $\eta$  is small enough.

We now want to determine how smooth  $u_i(t, z)$  are in  $z$ , first in the case  $i = 1$ . So pick points  $z$  and  $z'$  and set  $w = u_1(t, z) - u_1(t, z')$ . Hence

$$(B.18) \quad \Delta w = -R_z u_1(\cdot, z) + R_{z'} u_1(\cdot, z'), \quad w|_{\partial D} = 0,$$

or alternatively

$$(B.19) \quad (\Delta + R_z)w = -(R_z - R_{z'})u_1(\cdot, z'), \quad w|_{\partial D} = 0.$$

An argument similar to (B.11)–(B.14) yields, for  $s \in (0, r]$ ,

$$(B.20) \quad \|w\|_{C^{1+s}(\overline{D})} \leq C \|r^{ij}(\cdot, z) - r^{ij}(\cdot, z')\|_{C^s(\overline{D})} \|u_1(\cdot, z')\|_{C^{s+1}(\overline{D})}.$$

We already have a bound on  $u_1 = t_1 + v$  from (B.15). As for the other factor on the right side of (B.20), we can use the elementary estimate

$$(B.21) \quad \|f\|_{C^s(\overline{D})} \leq C \|f\|_{C^r(\overline{D})}^{s/r} \|f\|_{L^\infty(\overline{D})}^{1-s/r},$$

valid for  $s \in [0, r]$ , to deduce that

$$(B.22) \quad \|u_1(\cdot, z) - u_1(\cdot, z')\|_{C^{1+s}(\overline{D})} \leq C_s |z - z'|^{r-s}, \quad 0 < s \leq r,$$

given the latter alternative in hypothesis (B.1). The construction of  $u_2$  via (B.16) then yields

$$(B.23) \quad \|u_2(\cdot, z) - u_2(\cdot, z')\|_{C^{1+s}(\overline{D})} \leq C_s |z - z'|^{r-s}, \quad 0 < s \leq r.$$

Thus if we set

$$(B.24) \quad u_{ij} = \frac{\partial u_i}{\partial t_j}, \quad 1 \leq i, j \leq 2,$$

we have

$$(B.25) \quad \|u_{ij}(\cdot, z) - u_{ij}(\cdot, z')\|_{C^s(\overline{D})} \leq C_s |z - z'|^{r-s}, \quad 0 < s \leq r,$$

and hence, taking respectively  $s = r$  and  $s = \delta$ , close to 0, we have

$$(B.26) \quad \begin{aligned} |u_{ij}(t, z) - u_{ij}(t', z')| &\leq |u_{ij}(t, z) - u_{ij}(t', z)| + |u_{ij}(t', z) - u_{ij}(t', z')| \\ &\leq C|t - t'|^r + C_\delta|z - z'|^{r-\delta}. \end{aligned}$$

If we reverse the coordinate transformation (B.3), this estimate remains valid.

We obtain a CR function on an open set in  $U_1$  by composing  $u = u_1 + iu_2$  with the inverse of the bi-Lipschitz map  $G$ , given in (3.1):

$$(B.27) \quad \tilde{u} = u \circ G^{-1}.$$

We have

$$(B.28) \quad Y_j \tilde{u} = \frac{\partial u}{\partial t_j} \circ G^{-1}, \quad j = 1, 2,$$

where  $\{Y_1, Y_2\}$  is the Lipschitz frame field for  $\mathcal{E}$  that pulls back to  $\{\partial/\partial t_1, \partial/\partial t_2\}$ . It follows that

$$(B.29) \quad \tilde{u}, Y_j \tilde{u} \in C^{r-\delta}, \quad \forall \delta > 0.$$

While one cannot take  $\delta = 0$  in (B.26), one can improve the estimate, as follows. First, using

$$(B.30) \quad G : H^{-1,p}(D) \longrightarrow H^{1,p}(D), \quad 1 < p < \infty,$$

an argument parallel to (B.11)–(B.14) gives, in place of (B.20),

$$(B.31) \quad \begin{aligned} \|w\|_{H^{1,p}(D)} &\leq C\|(R_z - R_{z'})u_1(\cdot, z')\|_{H^{-1,p}(D)} \\ &\leq C\|r^{ij}(\cdot, z) - r^{ij}(\cdot, z')\|_{L^\infty(D)}\|u_1(\cdot, z')\|_{H^{1,p}(D)}. \end{aligned}$$

Then one can exploit the following local regularity result. Suppose  $\omega(h)$  is a modulus of continuity satisfying the Dini condition:

$$(B.32) \quad \int_0^{1/2} \frac{\omega(h)}{h} dh < \infty.$$

Then, with  $D_{1/2} = \{t : t_1^2 + t_2^2 < 1/4\}$ ,

$$(B.33) \quad \begin{aligned} \|w\|_{C^1(D_{1/2})} &\leq C\|w\|_{H^{1,p}(D)} + C\|(R_z - R_{z'})u_1(\cdot, z')\|_{C^{-1,\omega}(D)} \\ &\leq C\|w\|_{H^{1,p}(D)} + C\|(r^{ij}(\cdot, z) - r^{ij}(\cdot, z'))\partial_i u_1(\cdot, z')\|_{C^\omega(\overline{D})} \\ &\leq C\|w\|_{H^{1,p}(D)} + C\|r^{ij}(\cdot, z) - r^{ij}(\cdot, z')\|_{C^\omega(\overline{D})}\|u_1(\cdot, z')\|_{C^{r+1}(\overline{D})}. \end{aligned}$$

In view of (B.31), and the previous estimates on  $u_1(\cdot, z')$ , we have

$$(B.34) \quad \|u_1(\cdot, z) - u_1(\cdot, z')\|_{C^1(D_{1/2})} \leq C \|r^{ij}(\cdot, z) - r^{ij}(\cdot, z')\|_{C^\omega(\overline{D})}.$$

To estimate the right side of (B.34), we replace (B.21) by the following. Suppose

$$(B.35) \quad |f(x - y)| \leq C|x - y|^r, \quad |f(x) - f(y)| \leq C\delta.$$

Then

$$(B.36) \quad |f(x) - f(y)| \leq C\sigma_r(\delta)\omega(|x - y|),$$

where

$$(B.37) \quad \sigma_r(\delta) = \sup_{h \in (0, 1]} \frac{\min(\delta, h^r)}{\omega(h)}.$$

Of course, we pick  $\omega(h)$  decreasing to 0 as  $h \searrow 0$  more slowly than  $h^r$  for any  $r > 0$ , for example,

$$(B.38) \quad \omega(h) = \left(\log \frac{1}{h}\right)^{-1-a},$$

for some  $a > 0$ , so that  $h^r/\omega(h)$  is  $\nearrow$  on  $h \in (0, 1/2]$ . In such a case,

$$(B.39) \quad \sigma_r(\delta) \approx \frac{\delta}{\omega(\delta^{1/r})}.$$

We deduce that, under the hypothesis (B.1),

$$(B.40) \quad \begin{aligned} \|u_1(\cdot, z) - u_1(\cdot, z')\|_{C^1(D_{1/2})} &\leq C\sigma_r(|z - z'|^r) \\ &\leq C \frac{|z - z'|^r}{\omega(|z - z'|)}. \end{aligned}$$

Hence we can supplement (B.25) with

$$\|u_{ij}(\cdot, z) - u_{ij}(\cdot, z')\|_{C^0(D_{1/2})} \leq C \frac{|z - z'|^r}{\omega(|z - z'|)},$$

and improve (B.26) to

$$|u_{ij}(t, z) - u_{ij}(t', z')| \leq C|t - t'|^r + C \frac{|z - z'|^r}{\omega(|z - z'|)}.$$

This in turn leads to an improvement in the modulus of continuity in (B.29), to

$$(B.41) \quad |\tilde{u}(x) - \tilde{u}(x')|, |Y_j \tilde{u}(x) - Y_j \tilde{u}(x')| \leq C \frac{|x - x'|^r}{\omega(|x - x'|)}.$$

Next, we want to replace hypothesis (B.1) by

$$(B.42) \quad J \in \text{Lip}(U_1).$$

Then we replace  $C^r$  by  $\text{Lip}$  in (B.2), (B.5), and (B.8), and we supplement (B.13) by

$$(B.43) \quad G : C_*^0(\overline{D}) \longrightarrow C_*^2(\overline{D}).$$

Thus (B.14) is modified to

$$(B.44) \quad \|GR_z f\|_{C_*^2(\overline{D})} \leq C \|r^{ij}(\cdot, z)\|_{C_*^1(\overline{D})} \|f\|_{C_*^2(\overline{D})},$$

which leads to the existence of isothermal coordinates  $u_1, u_2$ , satisfying

$$(B.45) \quad \|u_i - t_i\|_{C_*^2(\overline{D})} \leq C\eta.$$

Analogues of (B.18)–(B.23) hold. We need to replace (B.21) by the interpolation inequality

$$(B.46) \quad \|f\|_{C_*^s(\overline{D})} \leq C \|f\|_{C_*^1(\overline{D})}^s \|f\|_{C_*^0(\overline{D})}^{1-s},$$

valid for  $s \in [0, 1]$ , and then we get

$$(B.47) \quad \|u_i(\cdot, z) - u_i(\cdot, z')\|_{C_*^{1+s}(\overline{D})} \leq C_s |z - z'|^{1-s}, \quad 0 < s \leq 1.$$

Keep in mind that  $C_*^{1+s}(\overline{D}) = C^{1+s}(\overline{D})$  for  $0 < s < 1$ . Similarly, in place of (B.25), we have

$$(B.48) \quad \|u_{ij}(\cdot, z) - u_{ij}(\cdot, z')\|_{C_*^s(\overline{D})} \leq C_s |z - z'|^{1-s}, \quad 0 < s \leq 1.$$

Consequently (B.26) is modified as follows. First, since elements of  $C_*^1(\overline{D})$  have a log-Lipschitz modulus of continuity, we have

$$(B.49) \quad |u_{ij}(t, z) - u_{ij}(t', z)| \leq C |t - t'| \log \frac{1}{|t - t'|},$$

for  $t, t' \in \overline{D}$ . On the other hand, (B.34) still applies, and we can take  $r = 1$  in (B.37) to obtain

$$(B.50) \quad \|u_{ij}(\cdot, z) - u_{ij}(\cdot, z')\|_{C^0(D_{1/2})} \leq C\sigma_1(|z - z'|),$$

where

$$(B.51) \quad \sigma_1(\delta) = \sup_{h \in (0, 1]} \frac{\min(\delta, h)}{\omega(h)}.$$

We thus obtain, in place of (B.40), the modulus of continuity estimate

$$(B.52) \quad |u_{ij}(t, z) - u_{ij}(t', z')| \leq C|t - t'| \log \frac{1}{|t - t'|} + C\sigma_1(|z - z'|),$$

for  $t, t' \in D_{1/2}$ . This in turn leads to

$$(B.53) \quad |\tilde{u}(x) - \tilde{u}(x')|, |Y_j \tilde{u}(x) - Y_j \tilde{u}(x')| \leq C\sigma^\#(|x - x'|),$$

where

$$(B.54) \quad \sigma^\#(\delta) = \max\left(\sigma_1(\delta), \delta \log \frac{1}{\delta}\right), \quad \text{for } 0 < \delta \leq 1.$$

If  $\omega(h)$  is given by (B.38), we have

$$(B.55) \quad \sigma^\#(\delta) = \delta \left(\log \frac{1}{\delta}\right)^{1+a}.$$

We formally state the main conclusion of this appendix. Since the result is local, we may as well take  $\Omega$  to be an open set in some Euclidean space.

**Proposition B.1.** *Let  $\Omega$  have a Lipschitz, Levi-flat CR-structure, with leaves tangent to  $\mathcal{E}$  of real dimension two. Then each  $p \in \Omega$  has a neighborhood  $U$  on which there is a CR-function*

$$(B.56) \quad \tilde{u} : U \longrightarrow \mathbb{C},$$

*which is a holomorphic diffeomorphism on each leaf, intersected with  $U$ , into  $\mathbb{C}$ , with the following regularity. For any  $a > 0$ , and any Lipschitz section  $Y$  of  $\mathcal{E}$ ,*

$$(B.57) \quad |\tilde{u}(x) - \tilde{u}(x')|, |Y \tilde{u}(x) - Y \tilde{u}(x')| \leq C_a |x - x'| \left(\log \frac{1}{|x - x'|}\right)^{1+a},$$

*for  $x, x' \in U$ ,  $|x - x'| \leq 1/2$ .*

REMARK. Since a tool in the analysis of the Lipschitz CR-structures was an analysis of families of much less regular almost complex structures, it is worth mentioning the fundamental work of Ahlfors and Bers [AB] on the endpoint case, involving merely  $L^\infty$  almost complex structures. See also [A] and [D] for treatments; the latter article also discusses dependence on parameters. In such a case the  $C^1$  regularity collapses to Hölder continuity, and it does not seem that techniques used there lead to an improvement of Proposition B.1.

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